1. Compute the following integrals:

\[(a) \int_{-\infty}^{\infty} \frac{\sin x}{x(x - \pi)} \, dx \quad (b) \frac{1}{2\pi i} \oint_C \frac{dz}{\sin(1/z)},\]

where \(C\) is the circle \(|z| = 1/5\) positively oriented.

\((15 \times 2 = 30\) points\)

2. Determine the automorphism group \(\text{Aut}(\mathbb{C})\) of the complex plane; i.e., the set of all one-to-one analytic maps of \(\mathbb{C}\) onto \(\mathbb{C}\). (Hint: examine the behavior at \(\infty\).)

\((20\) points\)

3. Let \(f\) be analytic on \(\mathbb{C} \setminus \{0\}\), and suppose that

\[f([|z| = 1]) \subseteq \mathbb{R} \quad \text{and} \quad f(z) = f(1/z) \text{ for all } z \neq 0.\]

Prove that \(f\) is real on \(\mathbb{R} \setminus \{0\}\).

\((20\) points\)

4. State whether each of the following statements is true or false. Give a short proof or a counterexample, as appropriate, in support of your claim.

\((10 \times 3 = 30\) points\)

(a) Let \(\Omega\) be an open subset of \(\mathbb{R}^2\) and let \(f : \Omega \to \mathbb{R}^2\) be a smooth map. Assume that \(f\) preserves orientation (i.e., the Jacobian of \(f\) is positive everywhere), and that \(f\) maps any pair of orthogonal curves to a pair of orthogonal curves. Then \(f\) must be holomorphic, after the usual identification of \(\mathbb{R}^2\) with \(\mathbb{C}\).

(b) There exists a non-polynomial entire function \(f\) such that the image of every unbounded sequence under \(f\) is some unbounded sequence.

(c) The only function that is analytic on the unit disc and satisfies

\[f'' \left( \frac{1}{p} \right) + f \left( \frac{1}{p} \right) = 0 \text{ for all prime integers } p\]

is the constant zero function.