9. \( f(x, y, z) = xyz, \ g(x, y, z) = x^2 + 2y^2 + 3z^2 - 6 \Rightarrow \nabla f = (yz, xz, xy), \ \lambda \nabla g = (2\lambda x, 4\lambda y, 6\lambda z). \) If \( \lambda = 0 \) then at least one of the coordinates is 0, in which case \( f(x, y, z) = 0. \) (None of these ends up giving a maximum or minimum.)

If \( \lambda \neq 0, \) then \( \nabla f = \lambda \nabla g \) implies \( \lambda = (yz)/(2x) = (xz)/(4y) = (xy)/(6z) \) or \( x^2 = 2y^2 \) and \( z^2 = \frac{3}{2}y^2. \) Thus \( x^2 + 2y^2 + 3z^2 - 6 \) implies \( 6y^2 = 6 \) or \( y = \pm 1. \) Thus the possible remaining points are \( (\sqrt{2}, \pm 1, \sqrt{\frac{3}{2}}), \) \( (\sqrt{2}, \pm 1, -\sqrt{\frac{3}{2}}), \) \( (-\sqrt{2}, \pm 1, \sqrt{\frac{3}{2}}), \) \( (-\sqrt{2}, \pm 1, -\sqrt{\frac{3}{2}}). \) The maximum value of \( f \) on the ellipsoid is \( \frac{3}{\sqrt{2}}, \) occurring when all coordinates are positive or exactly two are negative and the minimum is \( -\frac{3}{\sqrt{2}}, \) occurring when 1 or 3 of the coordinates are negative.

18. \( f(x, y) = 2x^2 + 3y^2 - 4x - 5 \Rightarrow \nabla f = (4x - 4, 6y) = (0, 0) \Rightarrow x = 1, y = 0. \) Thus \( (1, 0) \) is the only critical point of \( f, \) and it lies in the region \( x^2 + y^2 < 16. \) On the boundary, \( g(x, y) = x^2 + y^2 = 16 \Rightarrow \lambda \nabla g = (2\lambda x, 2\lambda y), \) so \( 6y = 2\lambda y \Rightarrow \) either \( y = 0 \) or \( \lambda = 3. \) If \( y = 0, \) then \( x = \pm 4; \) if \( \lambda = 3, \) then \( 4x - 4 = 2\lambda x \Rightarrow x = -2 \) and \( y = \pm 2 \sqrt{3}. \) Now \( f(1, 0) = -7, f(4, 0) = 11, f(-4, 0) = 43, \) and \( f(-2, \pm 2 \sqrt{3}) = 47. \) Thus the maximum value of \( f(x, y) \) on the disk \( x^2 + y^2 \leq 16 \) is \( f(-2, \pm 2 \sqrt{3}) = 47, \) and the minimum value is \( f(1, 0) = -7. \)

44. (a) By Theorem 15.6.15 [ET 14.6.15], the maximum value of the directional derivative occurs when \( \mathbf{u} \) has the same direction as the gradient vector.

(b) It is a minimum when \( \mathbf{u} \) is in the direction opposite to that of the gradient vector (that is, \( \mathbf{u} \) is in the direction of \( -\nabla f \)), since \( D_{\mathbf{u}} f = |\nabla f| \cos \theta \) (see the proof of Theorem 15.6.15 [ET 14.6.15]) has a minimum when \( \theta = \pi. \)

(c) The directional derivative is 0 when \( \mathbf{u} \) is perpendicular to the gradient vector, since then \( D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = 0. \)

(d) The directional derivative is half of its maximum value when \( D_{\mathbf{u}} f = |\nabla f| \cos \theta = \frac{1}{2} |\nabla f| \iff \cos \theta = \frac{1}{2} \iff \theta = \frac{\pi}{3}. \)
32. Because $X$ and $Y$ are independent, the joint density function for Xavier's and Yolanda's arrival times is the product of the individual density functions:

$$f(x, y) = f_1(x) f_2(y) = \begin{cases} \frac{1}{50} e^{-\frac{x}{50}} y & \text{if } x \geq 0, 0 \leq y \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

Since Xavier won't wait for Yolanda, they won't meet unless $X \geq Y$.

Additionally, Yolanda will wait up to half an hour but no longer, so they won't meet unless $X - Y \leq 30$. Thus the probability that they meet is $P((X, Y) \in D)$ where $D$ is the parallelogram shown in the figure. The integral is simpler to evaluate if we consider $D$ as a type II region, so

$$P((X, Y) \in D) = \int_0^{10} \int_{-e^{-y}}^{e^{-y}+30} \frac{1}{50} e^{-\frac{x}{50}} y \, dx \, dy = \int_0^{10} \left[ e^{-y} \right]_{-e^{-y}}^{e^{-y}+30} y \, dy = \int_0^{10} \left( 1 - e^{-y} \right) e^{-y} \, dy$$

By integration by parts (or Formula 96 in the Table of Integrals), this is

$$\frac{1}{50} (1 - e^{-30}) \left[ -(y + 1) e^{-y} \right]_0^{10} = \frac{1}{50} (1 - e^{-30})(1 - 11 e^{-10}) \approx 0.020.$$ This is only about a 2% chance they will meet.

Such is student life!

14. $\rho \leq 2$ represents the solid sphere of radius 2 centered at the origin. Notice that $x^2 + y^2 = (\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2 = \rho^2 \sin^2 \phi$. Then $\rho = \csc \phi \Rightarrow \rho \sin \phi = 1 \Rightarrow \rho^2 \sin^2 \phi = x^2 + y^2 = 1$, so $\rho \leq \csc \phi$ restricts the solid to that portion on or inside the circular cylinder

$$x^2 + y^2 = 1.$$

36. Place the center of the sphere at $(0, 0, 0)$, let the diameter of intersection be along the $z$-axis, one of the planes be the $xz$-plane and the other be the plane whose angle with the $xz$-plane is $\theta = \frac{\pi}{6}$. Then in spherical coordinates the volume is given by

$$V = \int_0^{\pi/6} \int_0^\pi \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{\pi}{6} \int_0^\pi \sin \phi \, d\phi \int_0^{\rho^2} \rho^2 \, d\rho = \frac{\pi}{6} \left( \frac{1}{3} \alpha^3 \right) = \frac{\pi}{18} \alpha^3.$$

44. The given integral is equal to

$$\lim_{R \to \infty} \int_0^R \rho^2 e^{-\rho^2} \sin \phi \, d\rho \
\phi \, d\theta = \lim_{R \to \infty} \left( \int_0^{\pi/2} \rho^2 e^{-\rho^2} \, d\phi \right) \left( \int_0^R \rho^2 e^{-\rho^2} \, d\rho \right)$$

Now use integration by parts with $u = \rho^2$, $dv = e^{-\rho^2} \, d\rho$ to get

$$\lim_{R \to \infty} 2\pi \left( \rho^2 \left( \frac{1}{2} e^{-\rho^2} \right) \right)_{\rho=R} - \int_0^R 2\rho \left( \frac{1}{2} e^{-\rho^2} \right) \, d\rho = \lim_{R \to \infty} 4\pi \left( \frac{1}{2} R^2 e^{-R^2} - \frac{1}{2} e^{-R^2} \right)_{\rho=R}$$

$$= 4\pi \lim_{R \to \infty} \left[ -\frac{1}{2} R^2 e^{-R^2} + \frac{1}{2} e^{-R^2} \right] = 4\pi \left( \frac{1}{2} \right) = 2\pi$$

(Note that $R^2 e^{-R^2} \to 0$ as $R \to \infty$ by l’Hospital’s Rule.)
Practice Problem Set 2 Solutions

14. \[ \int_0^1 \int_0^{\frac{1}{\sqrt{9}}} \frac{ye^{x^2}}{x^2} \, dy \, dx = \int_0^1 \int_0^{\frac{1}{\sqrt{9}}} \frac{ye^{x^2}}{x^2} \, dy \, dx = \int_0^1 \frac{e^{x^2}}{x^2} \left[ \frac{1}{2} y^2 \right]_{y=0}^{y=x^2} \, dx \]
   \[= \int_0^1 \frac{1}{2} xe^{x^2} \, dx = \frac{1}{4} e^{x^2} \bigg|_0^1 = \frac{1}{4}(e - 1) \]

21. \[ \iint_D \left( x^2 + y^2 \right)^{3/2} \, dA = \int_0^{\pi/3} \int_0^3 (r^2)^{3/2} r \, dr \, d\theta \]
   \[= \int_0^{\pi/3} d\theta \int_0^3 r^4 \, dr = \left[ \frac{1}{5} r^5 \right]_0^3 \]
   \[= \frac{\pi}{3} \cdot \frac{3^5}{5} = \frac{81\pi}{5} \]