Review Sheet 1

1. For what values of the number \( r \) is the function
   \[
   f(x, y, z) = \begin{cases} 
   \frac{(x+y+z)^r}{x^2+y^2+z^2} & \text{if } (x, y, z) \neq (0, 0, 0) \\
   0 & \text{if } (x, y, z) = 0
   \end{cases}
   \]
   continuous on \( \mathbb{R}^3 \)?
   (Answer: \( r > 2 \))

2. Among all planes that are tangent to the surface \( xy^2z^2 = 1 \), find the ones that are farthest from the origin.
   (Answer: \( (2^{2/5})x \pm (2^{9/10})y \pm (2^{9/10})z = 5 \))

3. Evaluate the integral
   \[
   \int_0^1 \int_0^1 e^{\max(x^2,y^2)} \, dy \, dx,
   \]
   where \( \max\{x^2, y^2\} \) means the larger of the two numbers \( x^2 \) and \( y^2 \).
   (Answer: \( e - 1 \))

4. If \( f : \mathbb{R} \to \mathbb{R} \) is continuous, show that
   \[
   \int_0^x \int_0^y f(t) \, dt \, dz = \frac{1}{2} \int_0^x (x-t)^2 f(t) \, dt.
   \]

5. Recall that a function \( f \) is harmonic if \( \nabla^2 f = 0 \).
   (a) Show that if \( f \) is a harmonic function in \( \mathbb{R}^2 \) then the line integral
   \[
   \int f_y \, dy - f_x \, dx
   \]
   is independent of path.
   (b) Show that for any harmonic function \( f \) in \( \mathbb{R}^2 \), \( \langle f_x, f_y \rangle \) and \( \langle f_y, -f_x \rangle \) form a pair of mutually orthogonal (i.e., perpendicular to each other) conservative vector fields.

6. (a) Sketch the curve \( C \) with parametric equations
   \[
   x = \cos t, \quad y = \sin t, \quad z = \sin t, \quad 0 \leq t \leq 2\pi.
   \]
   (b) Find \( \int_C 2xe^{2y} \, dx + (2x^2e^{2y} + 2y \cot z) \, dy - y^2 \csc^2 z \, dz \).
   (Answer: (a) an ellipse, (b) 0 )

7. Let
   \[
   \mathbf{F}(x, y) = \frac{1}{x^2 + y^2} \left[ (2x^3 + 2xy^2 - 2y)i + (2y^3 + 2x^2y + 2x)j \right].
   
   Evaluate \( \oint_C \mathbf{F} \cdot d\mathbf{r} \), where \( C \) is an arbitrary positively oriented simple closed curve containing the origin in its interior.
   (Answer: \( 4\pi \))

8. Find the positively oriented simple closed curve \( C \) for which the value of the line integral
   \[
   \int_C (y^3 - y) \, dx - 2x^3 \, dy
   \]
   (Answer: \( \frac{3}{2} \))
9. (a) As part of the lecture on div and curl, we reformulated Green’s theorem as follows:

\begin{equation}
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k} \, dA,
\end{equation}

where $C$ and $D$ satisfy the hypotheses of Green’s theorem. This led to a discussion prompted by two questions of Craig and Evan, about the significance of $\mathbf{k}$ in this formula, and possible generalizations of this result for curves $C$ not necessarily lying in the $(x, y)$ plane. We are now in a position to address this question in its entirety.

Let $C$ be a simple closed curve lying on a (not necessarily horizontal) plane $P$, and enclosing a domain $D$. Let $\mathbf{F}$ be a vector field in $\mathbb{R}^3$ with continuous partial derivatives on $D$. Find an identity similar to (1) that relates the line integral \( \oint_C \mathbf{F} \cdot d\mathbf{r} \) with an integral over the domain $D$. What has $k \, dA$ been replaced by?

(b) Let $C$ be a simple positively oriented closed curve lying in a plane with unit normal vector $\mathbf{n} = \langle a, b, c \rangle$. Show that the plane area enclosed by $C$ is

\[
\frac{1}{2} \oint_C (bz - cy)\,dx + (cx - az)\,dy + (ay - bx)\,dz.
\]