4. Since \((x + y + z)/(x^2 + y^2 + z^2)\) is a rational function with domain \(\{(x, y, z) \mid (x, y, z) \neq (0, 0, 0)\}\), \(f\) is continuous on \(\mathbb{R}^3\) if and only if \(\lim_{(x,y,z) \to (0,0,0)} f(x,y,z) = f(0,0,0) = 0\). Recall that \((a + b)^2 \leq 2a^2 + 2b^2\) and a double application of this inequality to \((x + y + z)\) gives \((x + y + z)^2 \leq 4x^2 + 4y^2 + 2z^2 \leq 4(x^2 + y^2 + z^2)\). Now for each \(r\),
\[
|\frac{(x+y+z)}{x^2+y^2+z^2}| = \left(\frac{(x+y+z)^2}{x^2+y^2+z^2}\right) = \left[4(x^2+y^2+z^2)\right]^{r/2} = 2^r(x^2+y^2+z^2)^{r/2}
\]
for \((x,y,z) \neq (0,0,0)\). Thus
\[
|f(x,y,z) - 0| = \left|\frac{(x+y+z)}{x^2+y^2+z^2}\right| \leq 2^r \frac{(x^2+y^2+z^2)^{r/2}}{x^2+y^2+z^2} = 2^r(x^2+y^2+z^2)^{(r/2)-1}
\]
for \((x,y,z) \neq (0,0,0)\). Thus if \((r/2) - 1 > 0\), that is \(r > 2\), then \(2^r(x^2+y^2+z^2)^{(r/2)-1} \to 0\) as \((x,y,z) \to (0,0,0)\) and so \(\lim_{(x,y,z) \to (0,0,0)} \frac{(x+y+z)}{x^2+y^2+z^2} = 0\). Hence for \(r > 2\), \(f\) is continuous on \(\mathbb{R}^3\). Now if \(r \leq 2\), then as \((x,y,z) \to (0,0,0)\) along the \(x\)-axis, \(f(x,0,0) = \frac{x}{x^2} = x^{-1}\) for \(x \neq 0\). So when \(r = 2\), \(f(x,y,z) \to 1 \neq 0\) as \((x,y,z) \to (0,0,0)\) along the \(x\)-axis and when \(r < 2\) the limit of \(f(x,y,z)\) as \((x,y,z) \to (0,0,0)\) along the \(x\)-axis doesn’t exist and thus can’t be zero. Hence for \(r \leq 2\) \(f\) isn’t continuous at \((0,0,0)\) and thus is not continuous on \(\mathbb{R}^3\).
8. The tangent plane to the surface \( xy^2 z^2 = 1 \), at the point \((x_0, y_0, z_0)\) is
\[
y^2 z_0^3 (x - x_0) + 2x_0 y_0 z_0^2 (y - y_0) + 2x_0 y_0^2 z_0 (z - z_0) = 0
\Rightarrow (y_0 z_0^2)x + (2x_0 y_0 z_0^2)y + (2x_0 y_0^2 z_0)z = 5x_0 y_0^2 z_0^2 = 5.
\]
Using the formula derived in Example 13.5.8 [ET 12.5.8], we find that the distance from \((0, 0, 0)\) to this tangent plane is
\[
D(x_0, y_0, z_0) = \frac{|5x_0 y_0^2 z_0^2|}{\sqrt{(y_0^2 z_0^2)^2 + (2x_0 y_0 z_0^2)^2 + (2x_0 y_0^2 z_0)^2}}.
\]
When \(D\) is a maximum, \(D^2\) is a maximum and \(\nabla D^2 = 0\). Dropping the subscripts, let
\[
f(x, y, z) = D^2 = \frac{25(xy^2)^2}{y^2 z^2 + 4x^2 z^2 + 4x^2 y^2}.
\]
Now use the fact that for points on the surface \(xy^2 z^2 = 1\) we have \(z^2 = \frac{1}{xy^2}\),
to get \(f(x, y) = D^2 = \frac{25x}{y^2 + 4x^2 + 4x^2 y^2} = \frac{25x^2 y^2}{y^2 + 4x^2 + 4x^2 y^2} = 0 \Rightarrow \frac{50x y^2 (y^2 + 4x^2 + 4x^2 y^2) - (8x + 12x^2 y^2) (25x^2 y^2)}{(y^2 + 4x^2 + 4x^2 y^2)^2} = 0 \Rightarrow xy^2 + 4x^2 + 4x^2 y^2 = 0 \Rightarrow xy^2 - 2x^2 y^4 = 0 \Rightarrow xy^2 (1 - 2x^2 y^2) = 0 \Rightarrow 1 = 2x^2 y^2 \quad \text{(since } x = 0, y = 0 \text{ both give a minimum distance of 0)}. \text{ Also } f_y = 0 \Rightarrow \frac{50x y^2 (y^2 + 4x^2 + 4x^2 y^2) - (2y + 12x^2 y^2) 25x^2 y^2}{(y^2 + 4x^2 + 4x^2 y^2)^2} = 0 \Rightarrow 4x^2 y - 4x^2 y^5 = 0 \Rightarrow x^2 y (1 - x^2 y^4) = 0 \Rightarrow x^2 y^4 = 1 \Rightarrow \text{Substituting } x = 1/y^2 \text{ into } 1 = 2y^2 x^2, \text{ we get } 1 = 2y^{-10} \Rightarrow y = \pm 2^{1/10} \Rightarrow x = 2^{-7/5} \Rightarrow z^2 = \frac{1}{xy^2} = \frac{1}{(2^{-2/5})(2^{1/5})} = 2^{1/5} \Rightarrow z = \pm 2^{1/10}.
\]
Therefore the tangent planes that are farthest from the origin are at the four points \((2^{-7/5}, \pm 2^{1/10}, \pm 2^{1/10})\). These points all give a maximum since the minimum distance occurs when \(x_0 = 0 \text{ or } y_0 = 0 \text{ in which case } D = 0\). The equations are
\[
(2^{1/10} 2^{1/5})x \pm [(2)(2^{2/5})(2^{1/6})] y \pm [(2)(2^{2/5})(2^{1/5})(2^{1/10})] z = 5 \Rightarrow (2^{3/5})x \pm (2^{9/10})y \pm (2^{9/10})z = 5.
\]

2. \[
\begin{array}{c}
\text{Let } R = \{(x, y) \mid 0 \leq x, y \leq 1\}. \text{ For } x, y \in R, \text{ max } \{x^2, y^2\} = x^2 \text{ if } x \geq y, \\
\text{and max } \{x^2, y^2\} = y^2 \text{ if } x \leq y. \text{ Therefore we divide } R \text{ into two regions:}
\end{array}
\]
\[
R = R_1 \cup R_2, \text{ where } R_1 = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x\} \text{ and}
\]
\[
R_2 = \{(x, y) \mid 0 \leq y \leq 1, 0 \leq x \leq y\}. \text{ Now max } \{x^2, y^2\} = x^2 \text{ for}
\]
\[
(x, y) \in R_1, \text{ and max } \{x^2, y^2\} = y^2 \text{ for } (x, y) \in R_2 \Rightarrow
\]
\[
\int_0^1 \int_0^1 e^{\max(x^2, y^2)} \, dy \, dx = \int_0^1 e^x \, dx = \int_0^1 e^{\max(x^2, y^2)} \, dy = \int_0^1 e^{\max(x^2, y^2)} \, dy + \int_0^1 e^{\max(y^2, x^2)} \, dx = \int_0^1 e^{x^2} \, dx + \int_0^1 e^{y^2} \, dy = \int_0^1 e^{x^2} \, dx + \int_0^1 ye^{y^2} \, dy = e^2 \left[\frac{e^{1/2}}{2} - 1\right] = e - 1.
\]
11. \[ \int_0^a \int_0^b \int_0^c f(t) \, dt \, dx \, dy = \iiint_E f(t) \, dV, \] where
\[ E = \{(t, z, y) \mid 0 \leq t \leq z, 0 \leq z \leq y, 0 \leq y \leq a\}. \]
If we let \( D \) be the projection of \( E \) on the \( yt \)-plane then
\[ D = \{(y, t) \mid 0 \leq t \leq a, t \leq y \leq x\}. \] And we see from the diagram
that \( E = \{(t, z, y) \mid t \leq z \leq y, 0 \leq y \leq x, 0 \leq t \leq a\}. \) So
\[
\int_0^a \int_0^b \int_0^c f(t) \, dt \, dx \, dy = \int_0^a \int_0^b \int_0^c f(t) \, dx \, dy \, dt = \int_0^a \left[ \int_0^b (y - t) f(t) \, dy \right] \, dt
\]
\[ = \int_0^a \left[ \left( \frac{1}{2} y^2 - ty \right) f(t) \right]_{y=0}^{y=b} \, dt = \int_0^a \left[ \frac{1}{2} x^2 - tx - \frac{1}{2} t^2 + t^2 \right] f(t) \, dt
\]
\[ = \int_0^a \left[ \frac{1}{2} x^2 - tx + \frac{1}{2} t^2 \right] f(t) \, dt = \int_0^a \left( \frac{1}{2} x^2 - 2tx + t^2 \right) f(t) \, dt
\]
\[ = \frac{1}{2} \int_0^a (x - t)^2 f(t) \, dt
\]

23. \( \nabla^2 f = 0 \) means \( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \). Now if \( F = f_y \mathbf{i} - f_x \mathbf{j} \) and \( C \) is any closed path in \( D \), then applying Green’s Theorem, we get
\[ \oint_C F \cdot d\mathbf{r} = \iint_D \left( \frac{\partial}{\partial x} \left( -f_x \right) - \frac{\partial}{\partial y} \left( f_y \right) \right) \, dA = -\iint_D (f_{xx} + f_{yy}) \, dA = -\iint_D 0 \, dA = 0
\]
Therefore the line integral is independent of path, by Theorem 17.3.3 [ET 16.3.3].

24. (a) \( x^2 + y^2 = \cos^2 t + \sin^2 t = 1 \), so \( C \) lies on the circular cylinder \( x^2 + y^2 = 1 \).

But also \( y = z \), so \( C \) lies on the plane \( y = z \). Thus \( C \) is the intersection of the plane \( y = z \) and the cylinder \( x^2 + y^2 = 1 \).

(b) Apply Stokes’ Theorem, \( \oint_C F \cdot d\mathbf{r} = \iint_S \text{curl} \, F \cdot dS \):
\[
\text{curl} \, F = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\partial/\partial x & \partial/\partial y & \partial/\partial z \\
2xe^{2y} & 2xe^{2y} & 2y \cot z
\end{vmatrix} = \left\langle -2y \csc^2 z, -2y \csc^2 z, 0 \right\rangle \cdot \left\langle 0, 4xe^{2y}, -4xe^{2y} \right\rangle = 0
\]
Therefore \( \oint_C F \cdot d\mathbf{r} = \iint_S 0 \cdot dS = 0 \).
38. Let $C'$ be the circle with center at the origin and radius $a$ as in the figure. Let $D$ be the region bounded by $C$ and $C'$. Then $D$'s positively oriented boundary is $C \cup (-C')$. Hence by Green's Theorem
\[
\int_C \mathbf{F} \cdot d\mathbf{r} + \int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = 0,
\]
so
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(r(t)) \cdot r'(t) \, dt.
\]
\[
= \int_0^{2\pi} \left[ \frac{2\alpha^2 \cos^2 t + 2\alpha^2 \cos t \sin^2 t - 2\alpha \sin t}{a^2} + \frac{2\alpha^2 \sin^2 t + 2\alpha^2 \cos^2 t \sin t + 2\alpha \cos t}{a^2} \right] \, dt
\]
\[
= \int_0^{2\pi} \frac{2\alpha^2}{a^2} \, dt = 4\pi.
\]

2. By Green's Theorem
\[
\int_C (y^2 - y) \, dx - 2x^2 \, dy = \iint_D \left[ \frac{\partial(-2x^2)}{\partial x} - \frac{\partial(y^2 - y)}{\partial y} \right] \, dA = \iint_D (1 - 6x^2 - 3y^2) \, dA.
\]
Notice that for $6x^2 + 3y^2 > 1$, the integrand is negative. The integral has maximum value if it is evaluated only in the region where the integrand is positive, which is within the ellipse $6x^2 + 3y^2 = 1$. So the simple closed curve that gives a maximum value for the line integral is the ellipse $6x^2 + 3y^2 = 1$.

3. The given line integral $\frac{1}{3} \int_C (bz - cy) \, dx + (cx - az) \, dy + (ay - bx) \, dz$ can be expressed as $\iint_C \mathbf{F} \cdot d\mathbf{r}$ if we define the vector field $\mathbf{F}$ by $\mathbf{F}(x, y, z) = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k} = \frac{1}{2}(bz - cy) \mathbf{i} + \frac{1}{2}(cx - az) \mathbf{j} + \frac{1}{2}(ay - bx) \mathbf{k}$. Then define $S$ to be the planar interior of $C$, so $S$ is an oriented, smooth surface. Stokes' Theorem says $\iint_S \text{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_C \mathbf{F} \cdot d\mathbf{r} = \iiint_S \text{curl} \mathbf{F} \cdot \mathbf{n} \, d\mathbf{S}$.

Now
\[
\text{curl} \mathbf{F} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}
\]
\[
= (\frac{1}{2}a + \frac{1}{2}a) \mathbf{i} + (\frac{1}{2}b + \frac{1}{2}b) \mathbf{j} + (\frac{1}{2}c + \frac{1}{2}c) \mathbf{k} = \alpha \mathbf{i} + \beta \mathbf{j} + \gamma \mathbf{k} = \mathbf{n}
\]
so $\text{curl} \mathbf{F} \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{n} = |\mathbf{n}|^2 = 1$, hence $\iint_S \text{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S d\mathbf{S}$ which is simply the surface area of $S$. Thus,
\[
\iint_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2} \int_C (bz - cy) \, dx + (cx - az) \, dy + (ay - bx) \, dz
\]
is the plane area enclosed by $C$.  

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