Math 217 Assignment 9
Due Friday November 27

■ **Problems to turn in:**

1. Use Green's theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where
$$\mathbf{F}(x, y) = y^2 \cos x\mathbf{i} + (x^2 + 2y \sin x)\mathbf{j}$$
and C is the triangle from $(0, 0)$ to $(2, 6)$ to $(2, 0)$ to $(0, 0)$.

2. (a) Let D denote the bounded region in the plane enclosed by the simple closed curve C. Show that
$$A(D) = \oint_C x dy.$$
Can you find another line integral over C which represents the same area?

   (b) Show that the coordinates of the centroid $(\bar{x}, \bar{y})$ of D are
$$\bar{x} = \frac{1}{2A(D)} \oint_C x^2 \, dy, \qquad \bar{y} = -\frac{1}{2A(D)} \oint_C y^2 dx.$$
where $A(D)$ denotes the area of the region D.

   (c) Use the formula in part (a) above to find the area under one arch of the cycloid $x = t - \sin t$, $y = 1 - \cos t$.

3. (a) Is the vector field
$$\mathbf{F}(x, y, z) = e^z\mathbf{i} + \mathbf{j} + xe^z\mathbf{k}$$
conservative?

   (b) Is there a vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that
$$\text{curl}(\mathbf{G}) = xyz\mathbf{i} - y^2z\mathbf{j} + yz^2\mathbf{k}?$$
Explain.

4. Is there a value of p for which $\text{div}(\mathbf{r}/|\mathbf{r}|^p) = 0$?

5. Let C and D be as in Problem 2, and let $\mathbf{n}$ denote the unit normal vector to C.

   (a) Starting with the statement of Green's theorem, establish the identity
$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \text{div}\mathbf{F}(x, y) \, dA.$$

   (b) Let f and g be functions with continuous partial derivatives in D. Setting $\nabla^2 = \nabla \cdot \nabla$ and using the identity in part (a), prove Green's first identity
$$\iint_D f\nabla^2 g \, dA = \oint_C f(\nabla g) \cdot \mathbf{n} \, ds - \iint_D \nabla f \cdot \nabla g dA.$$

6. Identify the surfaces
$$(a) \quad \mathbf{r}(u, v) = 2 \sin u\mathbf{i} + 3 \cos u\mathbf{j} + v\mathbf{k}, \; 0 \le v \le 2,$$
$$(b) \quad \mathbf{r}(s, t) = \langle s, t, t^2 - s^2 \rangle.$$

7. Find the area of the part of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices $(0, 0)$, $(0, 1)$ and $(2, 1)$.
8. Find the area of the part of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$.