34. $z = f(x, y) = 1000 - 0.005x^2 - 0.01y^2 \quad \Rightarrow \quad \nabla f(x, y) = \langle -0.01x, -0.02y \rangle$ and $\nabla f(60, 40) = \langle -0.6, -0.8 \rangle$.

(a) Due south is in the direction of the unit vector $\mathbf{u} = -\mathbf{j}$ and

$D_{\mathbf{u}} f(60, 40) = \nabla f\,(60, 40) \cdot \langle 0, -1 \rangle = \langle -0.6, -0.8 \rangle \cdot \langle 0, -1 \rangle = 0.8$. Thus, if you walk due south from $(60, 40, 966)$ you will ascend at a rate of 0.8 vertical meters per horizontal meter.

(b) Northwest is in the direction of the unit vector $\mathbf{u} = \frac{1}{\sqrt{2}} \langle -1, 1 \rangle$ and

$D_{\mathbf{u}} f(60, 40) = \nabla f\,(60, 40) \cdot \frac{1}{\sqrt{2}} \langle -1, 1 \rangle = \langle -0.6, -0.8 \rangle \cdot \frac{1}{\sqrt{2}} \langle -1, 1 \rangle = -\frac{0.2}{\sqrt{2}} \approx -0.14$. Thus, if you walk northwest from $(60, 40, 966)$ you will descend at a rate of approximately 0.14 vertical meters per horizontal meter.

(c) $\nabla f(60, 40) = \langle -0.6, -0.8 \rangle$ is the direction of largest slope with a rate of ascent given by

$|\nabla f(60, 40)| = \sqrt{(-0.6)^2 + (-0.8)^2} = 1$. The angle above the horizontal in which the path begins is given by

$\tan\theta = 1 \quad \Rightarrow \quad \theta = 45\,°$.

44. $F(x, y, z) = yz - \ln(x + z) \quad \Rightarrow \quad \nabla F(x, y, z) = \left\langle -\dfrac{1}{x+z}, z, y - \dfrac{1}{x+z} \right\rangle$ and $\nabla F(0, 0, 1) = \langle -1, 1, -1 \rangle$.

(a) $(-1)(x - 0) + (1)(y - 0) - 1(z - 1) = 0$ or $x - y + z = 1$

(b) Parametric equations are $x = -t,\ y = t,\ z = 1 - t$ and symmetric equations are $\dfrac{x}{-1} = \dfrac{y}{1} = \dfrac{z-1}{-1}$ or $-x = y = 1 - z$.

44. Let x, y, z, be the positive numbers. Then $x + y + z = 12$ and we want to minimize $x^2 + y^2 + z^2 = x^2 + y^2 + (12 - x - y)^2 = f(x, y)$ for $0 < x, y < 12$. $f_x = 2x + 2(12 - x - y)(-1) = 4x + 2y - 24$, $f_y = 2y + 2(12 - x - y)(-1) = 2x + 4y - 24$, $f_{xx} = 4$, $f_{xy} = 2$, $f_{yy} = 4$. Then $f_x = 0$ implies $4x + 2y = 24$ or $y = 12 - 2x$ and substituting into $f_y = 0$ gives $2x + 4(12 - 2x) = 24 \quad \Rightarrow \quad 6x = 24 \quad \Rightarrow \quad x = 4$ and then $y = 4$, so the only critical point is $(4, 4)$. $D(4, 4) = 16 - 4 > 0$ and $f_{xx}(4, 4) = 4 > 0$, so $f(4, 4)$ is a local minimum. $f(4, 4)$ is also the absolute minimum [compare to the values of f as $x, y \to 0$ or 12] so the numbers are $x = y = z = 4$.

56. Any such plane must cut out a tetrahedron in the first octant. We need to minimize the volume of the tetrahedron that passes through the point $(1, 2, 3)$. Writing the equation of the plane as $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$, the volume of the tetrahedron is given by $V = \dfrac{abc}{6}$. But $(1, 2, 3)$ must lie on the plane, so we need $\dfrac{1}{a} + \dfrac{2}{b} + \dfrac{3}{c} = 1 \quad (\star)$ and thus can think of c as a function of a and b. Then $V_a = \dfrac{b}{6}\left(c + a\dfrac{\partial c}{\partial a}\right)$ and $V_b = \dfrac{a}{6}\left(c + b\dfrac{\partial c}{\partial b}\right)$. Differentiating $(\star)$ with respect to a we get $-a^{-2} - 3c^{-2}\dfrac{\partial c}{\partial a} = 0 \quad \Rightarrow$ $\dfrac{\partial c}{\partial a} = \dfrac{-c^2}{3a^2}$, and differentiating $(\star)$ with respect to b gives $-2b^{-2} - 3c^{-2}\dfrac{\partial c}{\partial b} = 0 \quad \Rightarrow \quad \dfrac{\partial c}{\partial b} = \dfrac{-2c^2}{3b^2}$. Then $V_a = \dfrac{b}{6}\left(c + a\dfrac{-c^2}{3a^2}\right) = 0 \quad \Rightarrow \quad c = 3a$, and $V_b = \dfrac{a}{6}\left(c + b\dfrac{-2c^2}{3b^2}\right) = 0 \quad \Rightarrow \quad c = \frac{3}{2}b$. Thus $3a = \frac{3}{2}b$ or $b = 2a$. Putting these into $(\star)$ gives $\frac{3}{a} = 1$ or $a = 3$ and then $b = 6$, $c = 9$. Thus the equation of the required plane is $\frac{x}{3} + \frac{y}{6} + \frac{z}{9} = 1$ or $6x + 3y + 2z = 18$.

41. We need to find the extreme values of $f(x,y,z) = x^2 + y^2 + z^2$ subject to the two constraints $g(x,y,z) = x + y + 2z = 2$ and $h(x,y,z) = x^2 + y^2 - z = 0$. $\nabla f = \langle 2x, 2y, 2z \rangle$, $\lambda \nabla g = \langle \lambda, \lambda, 2\lambda \rangle$ and $\mu \nabla h = \langle 2\mu x, 2\mu y, -\mu \rangle$. Thus we need $2x = \lambda + 2\mu x$ **(1)**, $2y = \lambda + 2\mu y$ **(2)**, $2z = 2\lambda - \mu$ **(3)**, $x + y + 2z = 2$ **(4)**, and $x^2 + y^2 - z = 0$ **(5)**.
From **(1)** and **(2)**, $2(x - y) = 2\mu(x - y)$, so if $x \neq y$, $\mu = 1$. Putting this in **(3)** gives $2z = 2\lambda - 1$ or $\lambda = z + \frac{1}{2}$, but putting $\mu = 1$ into **(1)** says $\lambda = 0$. Hence $z + \frac{1}{2} = 0$ or $z = -\frac{1}{2}$. Then **(4)** and **(5)** become $x + y - 3 = 0$ and $x^2 + y^2 + \frac{1}{2} = 0$. The last equation cannot be true, so this case gives no solution. So we must have $x = y$. Then **(4)** and **(5)** become $2x + 2z = 2$ and $2x^2 - z = 0$ which imply $z = 1 - x$ and $z = 2x^2$. Thus $2x^2 = 1 - x$ or $2x^2 + x - 1 = (2x - 1)(x + 1) = 0$ so $x = \frac{1}{2}$ or $x = -1$. The two points to check are $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $(-1, -1, 2)$: $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{3}{4}$ and $f(-1, -1, 2) = 6$. Thus $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is the point on the ellipse nearest the origin and $(-1, -1, 2)$ is the one farthest from the origin.

56. Inside D: $f_x = 2xe^{-x^2-y^2}(1 - x^2 - 2y^2) = 0$ implies $x = 0$ or $x^2 + 2y^2 = 1$. Then if $x = 0$, $f_y = 2ye^{-x^2-y^2}(2 - x^2 - 2y^2) = 0$ implies $y = 0$ or $2 - 2y^2 = 0$ giving the critical points $(0, 0)$, $(0, \pm 1)$. If $x^2 + 2y^2 = 1$, then $f_y = 0$ implies $y = 0$ giving the critical points $(\pm 1, 0)$. Now $f(0, 0) = 0$, $f(\pm 1, 0) = e^{-1}$ and $f(0, \pm 1) = 2e^{-1}$. On the boundary of D: $x^2 + y^2 = 4$, so $f(x, y) = e^{-4}(4 + y^2)$ and f is smallest when $y = 0$ and largest when $y^2 = 4$. But $f(\pm 2, 0) = 4e^{-4}$, $f(0, \pm 2) = 8e^{-4}$. Thus on D the absolute maximum of f is $f(0, \pm 1) = 2e^{-1}$ and the absolute minimum is $f(0, 0) = 0$.

60. $f(x,y) = 1/x + 1/y$, $g(x,y) = 1/x^2 + 1/y^2 = 1 \quad \Rightarrow \quad \nabla f = \left\langle -x^{-2}, -y^{-2} \right\rangle = \lambda \nabla g = \left\langle -2\lambda x^{-3}, -2\lambda y^{-3} \right\rangle$. Then $-x^{-2} = -2\lambda x^3$ or $x = 2\lambda$ and $-y^{-2} = -2\lambda y^{-3}$ or $y = 2\lambda$. Thus $x = y$, so $1/x^2 + 1/y^2 = 2/x^2 = 1$ implies $x = \pm\sqrt{2}$ and the possible points are $\left(\pm\sqrt{2}, \pm\sqrt{2}\right)$. The absolute maximum of f subject to $x^{-2} + y^{-2} = 1$ is then $f\left(\sqrt{2}, \sqrt{2}\right) = \sqrt{2}$ and the absolute minimum is $f\left(-\sqrt{2}, -\sqrt{2}\right) = -\sqrt{2}$.