Problem 1. Let \((\Omega, d)\) be a metric space, \(G \subseteq \mathbb{C}\) an open set, and \(\rho\) the metric we have defined in class on the space of continuous functions \(C(G, \Omega)\). Prove the following facts about \(\rho\):

(a) If \(\epsilon > 0\) is given then there is a \(\delta > 0\) and a compact set \(K \subset G\) such that for any \(f, g \in C(G, \Omega)\)

\[
\sup_{z \in K} d(f(z), g(z)) < \delta \implies \rho(f, g) < \epsilon.
\]

(b) Conversely, if \(\delta > 0\) and a compact set \(K\) are given, there is an \(\epsilon > 0\) such that for \(f, g \in C(g, \Omega)\),

\[
\rho(f, g) < \epsilon \implies \sup_{z \in K} d(f(z), g(z)) < \delta.
\]

(c) Use parts (a) and (b) to obtain the following characterization of open sets in the topology \((C(G, \Omega), \rho)\): A set \(O \subset C(G, \Omega)\) is open if and only if for each \(f \in O\) there is a compact set \(K\) and a \(\delta > 0\) such that

\[
\{g : d(f(z), g(z)) < \delta, z \in K\} \subset O.
\]

Problem 2. Recall that the definition of the metric \(\rho\) on \(C(G, \Omega)\) involved a sequence \(\{K_n : n \geq 1\}\) of compact subsets of \(G\). Show that the topology on \(C(G, \Omega)\) generated by \(\rho\) is invariant under the choice of these sets.

Problem 3. Show that \((C(G, \Omega), \rho)\) is a complete metric space if and only if \((\Omega, d)\) is complete.

Problem 4. Suppose that \(\mathcal{F} \subseteq C(G, \Omega)\) is equicontinuous at each point of \(G\); then show that \(\mathcal{F}\) is equicontinuous over each compact subset of \(G\).

Problem 5. Let \((X_n, d_n)\) be a metric space for each \(n \geq 1\) and let \(X = \prod_{n=1}^{\infty} X_n\) be their Cartesian product. That is, \(X = \{x_n : x_n \in X_n\} \text{ for each } n \geq 1\). For \(\xi = \{x_n\}\) and \(\eta = \{y_n\}\) in \(X\) define

\[
d(\xi, \eta) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}.
\]

(a) Verify that \((X, d)\) as defined above is a metric space.
(b) Show that convergence in the topology of $X$ is equivalent to coordinatewise convergence.

(c) If $(X_n, d_n)$ is compact for every $n \geq 1$, then $X$ is compact.

**Problem 6.** This exercise shows how the mean square convergence dominates the uniform convergence of analytic functions. If $U$ is an open subset of $\mathbb{C}$ we use the notation

$$
||f||_{L^2(U)} = \left[ \int_U |f(z)|^2 \, dxdy \right]^{\frac{1}{2}}
$$

for the mean square norm, and

$$
||f||_{L^\infty(U)} = \sup_{z \in U} |f(z)|
$$

for the sup norm.

(a) If $f$ is holomorphic in a neighborhood of the disk $B(z_0; r)$, show that for any $0 < s < r$ there exists a constant $C_{r,s}$ (depending only on $r$ and $s$ but independent of $f$ or $z_0$) such that

$$
||f||_{L^\infty(B(z_0; s))} \leq C_{r,s} ||f||_{L^2(B(z_0; r))}.
$$

(b) Prove that if $\{f_n\}$ is a Cauchy sequence of holomorphic functions in the mean square norm $|| \cdot ||_{L^2(U)}$, then the sequence $\{f_n\}$ converges uniformly on every compact subset $U$ to a holomorphic function.

**Problem 7.**

(a) Let $X$ and $\Omega$ be metric spaces and suppose that $f : X \to \Omega$ is one-one and onto. Show that $f$ is an open map if and only if $f$ is a closed map.

(b) Let $P : \mathbb{C} \to \mathbb{R}$ be defined by $P(z) = \text{Re}(z)$. Show that $P$ is an open map but not closed.

**Problem 8.** It is easy to see that pointwise convergence of a sequence of continuous functions is in general a weaker condition than convergence of compacta. However, under certain additional hypotheses, the former implies the latter. This problem illustrates two such examples.

(a) Consider $C(G, \mathbb{R})$ and suppose that $\{f_n\}$ is a sequence in $C(G, \mathbb{R})$ which is monotonically increasing (i.e., $f_n(z) \leq f_{n+1}(z)$ for all $z \in G$) and satisfies

$$
\lim_{n \to \infty} f_n(z) = f(z) \quad \text{for all } z \in G \text{ where } f \in C(G, \mathbb{R}).
$$

Show that $f_n \overset{L^1}{\to} f$. This result is due to Dini.
(b) Let \( \{f_n\} \subset C(G, \Omega) \) and suppose that \( \{f_n\} \) is equicontinuous. If \( f \in C(G, \Omega) \) and \( f_n(z) \to f(z) \) for each \( z \) then show that \( f_n \xrightarrow{p} f \).

**Problem 9.** Let \( f \) be analytic on the unit disk with series expansion \( \sum_{n \geq 0} a_n z^n \) at 0. Let \( \mu = \inf \{|f(z)| : |z| = 1\} \), and suppose that \( f \) has at most \( m \) zeroes in \( B(0; 1) \). Prove that
\[
\mu \leq |a_0| + \cdots + |a_m|.
\]

**Problem 10.** Let \( F \subseteq C(G, \mathbb{C}) \), where \( G \) is an open subset of \( \mathbb{C} \). If \( F \) is pointwise bounded, i.e.,
\[
\sup\{|f(z)| : f \in F\} < \infty \quad \text{for all } z \in G,
\]
does this imply that \( F \) is locally bounded? Would your answer change if \( F \) was required to be a subset of \( H(G) \)?

(*Hint:* For the second part of the question, argue that it suffices to create a sequence of analytic functions that converges pointwise to a non-analytic function. Then construct such a sequence of analytic functions (in fact a sequence of polynomials), taking for granted the following fact: *Let \( K \) be a compact subset of the complex plane with connected complement. Let \( f \) be a function analytic in a neighborhood of \( K \). Then there is a sequence of polynomials which converges to \( f \) uniformly on \( K \).* Choose a sequence of compact (but not necessarily connected) sets \( K_n \) gradually filling up \( \mathbb{C} \), and a sequence of functions \( g_n \) analytic on an open neighborhood of \( K_n \) on which the statement above may be applied.)