Problems to turn in:
(1) In each case sketch the region and then compute the volume of the solid region.

(a) The “ice-cream cone” region which is bounded above by the hemisphere \( z = \sqrt{a^2 - x^2 - y^2} \) and below by the cone \( z = \sqrt{x^2 + y^2} \).

Solution. In spherical coordinates,
\[
V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^a \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 2\pi \int_0^{\pi/4} \sin \phi \rho^3 / 3 \bigg|_0^a \, d\phi \\
= 2\pi (-\cos \phi) \bigg|_0^{\pi/4} a^3 / 3 = \frac{2\pi a^3}{3}(-\cos \frac{\pi}{4} + \cos 0) = \frac{\pi a^3(2 - \sqrt{2})}{3}
\]
or in cylindrical coordinates,
\[
V = \int_0^{2\pi} \int_0^{a/\sqrt{2}} \int_{-\sqrt{a^2 - r^2}}^{\sqrt{a^2 - r^2}} r \, dz \, dr \, d\theta = 2\pi \int_0^{a/\sqrt{2}} (r\sqrt{a^2 - r^2} - r^2) \, dr \\
= 2\pi \left[ \frac{-a^2 - r^2}{3} \right]_0^{a/\sqrt{2}} = 2\pi \frac{-a^2 - a^2/2}{3} + (a^2 - 0^2)^{3/2} - (a/\sqrt{2})^3 (0)^3 \\
= 2\pi \frac{a^3 - 2a^3/2\sqrt{2}}{3} = \frac{\pi a^3(2 - \sqrt{2})}{3}.
\]

(b) The region bounded by \( z = x^2 + 3y^2 \) and \( z = 4 - y^2 \).

Solution. The parabolic cylinder \( z = 4 - y^2 \) comprises the top of the surface (considered in terms of \( z \)) and the paraboloid \( z = x^2 + 3y^2 \) is the bottom surface in terms of \( z \). To determine the region of the \( xy \)-plane which the region bounded by these two surfaces lies over, we intersect the two surfaces, in this case we can set them equal to each other. We see that \( x^2 + 3y^2 = 4 - y^2 \) if and only if \( x^2 + 4y^2 = 4 \) if and only if \( (x/2)^2 + y^2 = 1 \). We will set up and compute the integral of this volume in rectangular coordinates (using a table of integrals to compute the anti-derivative \( \int (4 - x^2)3/2 \, dx \)).

\[
V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} (4-y^2) \, dy \, dx \\
= \int_{-2}^{2} \left[ (4 - x^2)y - \frac{(4/3)y^3}{2} \right]_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} \, dx \\
= 2 \int_{-2}^{2} \left( \frac{(4 - x^2)^{3/2}}{2} - \frac{4 - x^2)^{3/2}}{6} \right) \, dx \\
= \frac{2}{3} \int_{-2}^{2} (4 - x^2)^{3/2} \, dx \\
= \frac{2}{3} \left[ \frac{x}{8} \left( 5 - 2x^2 - 2x^2 \right) \sqrt{4 - x^2} + \frac{3 \cdot 2^4}{8} \sin^{-1}(x/2) \right]_{-2}^{2} \\
= 4(\sin^{-1}(1) - \sin^{-1}(-1)) = 4\pi
\]

Another way to compute this integral would be to make a substitution \( x = 2u \), so \( dx = 2du \) and we would integrate over a circle of radius 1 in \((u, y)\), which we will call \( \tilde{R} \) whereas the ellipse will be called \( R \). This makes everything much simpler. Let’s see what happens.
\[ V = \int \int _R \left( \int _{x^2+3y^2} ^{4-y^2} dz \right) dA = \int \int _R (4 - x^2 - 4y^2) dxdy = \int \int _R (4 - 4u^2 - 4y^2) 2dudy \]
\[ = \int _0 ^{2\pi} \int _0 ^1 (4 - 4r^2) 2rdrd\theta = \int _0 ^{2\pi} d\theta \int _0 ^1 (8r - 8r^3) dr = 2\pi [4r^2 - 2r^4]_0 ^1 = 2\pi (4 - 2) = 4\pi \]

(c) A sphere with a cylindrical hole bored through its centre. Specifically, the region inside the sphere \( x^2 + y^2 + z^2 = 9 \) and outside the cylinder \( x^2 + y^2 = 4 \).

**Solution.** A sphere of radius 3 has volume \( V_S = 36\pi \). Let \( V_C \) denote the volume inside the given sphere and the given cylinder simultaneously. The volume we want, \( V = V_S - V_C \).

Let’s compute \( V_C \) using cylindrical coordinates.

\[ V_C = \int _0 ^{2\pi} \int _0 ^2 \int _0 ^{(9-r^2)^{1/2}} r dzdrd\theta = \int _0 ^{2\pi} \int _0 ^2 (9 - r^2)^{1/2} 2rdrd\theta \]
\[ = 2\pi \left[ \frac{-2}{3} (9 - r^2)^{3/2} \right] ^2 _0 = \frac{4\pi}{3} (9^{3/2} - 5^{3/2}) = 36\pi - \frac{4\pi 5^{3/2}}{3}; \]

hence \( V = V_S - V_C = \frac{4\pi 5^{3/2}}{3} \).

(2) Switch these integrals to spherical coordinates and compute:

**Solution.** \( I_1 \) is an integral over the top half of a solid sphere of radius 3, centred at the origin.

\[ I_1 = \int _{-\pi/2} ^{\pi/2} \int _{\phi=0} ^{2\pi} \int _{\rho=0} ^{r^3} [\rho \cos \phi] \sqrt{\rho^2 \rho^2 \sin^2 \phi} d\rho d\theta d\phi \]
\[ = \left( \int _{\phi=0} ^{\pi/2} \sin \phi \cos \phi d\phi \right) \left( \int _{\theta=0} ^{2\pi} d\theta \right) \left( \int _{\rho=0} ^{3} \rho^4 d\rho \right) \]
\[ = \left[ \frac{1}{2} \sin^2 \phi \right] ^{\pi/2} _0 (2\pi) \left( \frac{3^5}{5} \right) = \frac{243\pi}{5} \]

\( I_2 \) is a solid region contained within \( x > 0, y > 0, z > 0 \). The solid is above the cone \( z = \sqrt{x^2 + y^2} \) and below the sphere \( x^2 + y^2 + z^2 = 18 \). To check this, note that the cone meets the sphere at the height where \( z^2 + z^2 = 18 \), \( z = 3 \), and the ring where they intersect is \( x^2 + y^2 = 9 \). The angle of the point of the bottom of the cone is \( \phi = \pi/4 \). Putting this together, we have
\[ I_2 = \int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2 + y^2 + z^2) \, dz \, dy \, dx \]

\[ = \int_0^{\pi/4} \int_0^{\pi/2} \int_0^{\sqrt{18}} (\rho^2) \, \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \]

\[ = \left( \int_0^{\pi/4} \sin \phi \, d\phi \right) \left( \int_0^{\pi/2} d\theta \right) \left( \int_0^{\sqrt{18}} \rho^4 \, d\rho \right) \]

\[ = \left[ -\cos \phi \right]_0^{\pi/4} \left( \frac{\pi}{2} \right) \left( \frac{(3\sqrt{2})^5}{5} \right) = \frac{486\pi}{5} (\sqrt{2} - 1) \]

(3) Calculate the moment of inertia of a circular pipe of outer radius \( a \), inner radius \( b \), length \( L \) and uniform density \( R \), rotating about its centre axis. From your answer, let \( b \to 0 \) and derive the formula for a solid cylinder too.

\textit{Solution.} Line the cylinder up along the \( z \)-direction and then the integral we need is easy to do in cylindrical coordinates:

\[ \int \int \int (x^2 + y^2) RdV = R \int_0^{2\pi} \int_0^{\pi/2} \int_b^a \rho^2 r \, d\rho \, dr \, d\theta = \frac{2}{5} \pi LR(a^4 - b^4). \]

Letting \( b \to 0 \), we obtain the moment of inertia of a solid cylinder, \((2/5)\pi LRa^4\).

(4) Find the gradient vector field of \( f(x, y) = \sqrt{x^2 + y^2} \) and \( g(x, y) = x^2 - y \). In each case, plot the gradient vector field and the contour plot of the function, on the same diagram.

\[ \nabla f = \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right), \quad \nabla g = (2x, -1) \]

(5) Compute \( \int_C f(x, y, z) \, ds \) for the following curves and functions.

(a) \( C_1 : \mathbf{r}(t) = (30 \cos^3 t, 30 \sin^3 t) \) for \( 0 \leq t \leq \pi/2 \) and \( f(x, y) = 1 + y/3 \).

\textit{Solution.} First, \( ds = |\mathbf{r}'(t)| \, dt = \sqrt{(-90 \cos^2 t \sin t)^2 + (90 \sin^2 t \cos t)^2} \, dt = 90 \cos t \sin t \, dt \). Now we are in a position to compute the line integral.
\[
\int_C (1 + y/3)ds = \int_0^{\pi/2} (1 + 10 \sin^3 t) 90 \cos t \sin t dt = \int_0^{\pi/2} (90 \sin t + 900 \sin^4 t) \cos t dt
\]
\[
= \int_{u=0}^1 (90u + 900u^4)du, \text{ where } u = \sin t
\]
\[
= [45u^2 + 180u^5]_0^1 = 225
\]

(b) \(C_2 : \mathbf{r}(t) = (t^2/2, t^3/3)\) for \(0 \leq t \leq 1\) and \(f(x, y) = x^2 + y^2\).

\textbf{Solution.} Again we start by computing \(ds = |\mathbf{r}'(t)|dt = t\sqrt{1+t^2}dt.\) Then
\[
\int_C (x^2 + y^2)ds = \int_0^1 ((t^2/2)^2 + (t^3/3)^2)t\sqrt{1+t^2}dt = \frac{1}{4} \int_0^1 t^4 \sqrt{1+t^2}(tdt) + \frac{1}{9} \int_0^1 t^6 \sqrt{1+t^2}(tdt)
\]
\[
= \frac{1}{8} \int_{u=1}^2 (u - 1)^2 \sqrt{u}du + \frac{1}{18} \int_{u=1}^2 (u - 1)^3 \sqrt{u}du, \text{ where } u = 1 + t^2
\]
\[
= \frac{1}{2} \int_1^2 (u^{5/2} - 2u^{3/2} + u^{1/2})du + \frac{1}{18} \int_1^2 (u^{7/2} - 3u^{5/2} + 3u^{3/2} - u^{1/2})du
\]
\[
= \left[ \frac{u^{7/2}}{28} - \frac{u^{5/2}}{10} + \frac{u^{3/2}}{12} + \frac{u^{9/2}}{81} - \frac{u^{7/2}}{21} + \frac{u^{5/2}}{15} - \frac{u^{3/2}}{27} \right]_1^2
\]
\[
= \left[ \frac{u^{9/2}}{81} - \frac{u^{7/2}}{84} - \frac{u^{5/2}}{30} + \frac{5u^{3/2}}{108} \right]_1^2
\]
\[
= (2^{9/2}/81 - 2^{7/2}/84 - 2^{5/2}/30 + 5 \cdot 2^{3/2}/108) - (1/81 - 1/84 - 1/30 + 5/108)
\]

(c) \(C_3 : \mathbf{r}(t) = (1, t^2, t^3)\) for \(0 \leq t \leq 1\) and \(f(x, y, z) = e^{\sqrt{z}}.\)

\textbf{Solution.}
\[
\int_C e^{\sqrt{z}}ds = \int_0^1 e^\sqrt{0^2 + 0^2 + (2t)^2}dt = \int_0^1 2te^t dt = |2te^t - 2e^t|_0^1 = 2
\]

Note that we had to integrate by parts to anti-differentiate \(2te^t.\) (You let \(u = 2t\) and \(dv = e^t.\))

(6) Determine whether or not the following vector fields are conservative. In the cases where \(\mathbf{F}\) is conservative, find a function \(\varphi\) such that \(\mathbf{F}(x, y, z) = \nabla \varphi(x, y, z).\)

(a) \(\mathbf{F} = (2xy + z^2)\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 2xz)\mathbf{k}.\)

\textbf{Solution.} We first test to determine whether or not \(\mathbf{F}\) might be conservative. Letting \(F_1 = 2xy + z^2, F_2 = x^2 + 2yz,\) and \(F_3 = y^2 + 2xy\) (as usual), it is easy to verify that \(\partial F_1/\partial y = \partial F_2/\partial x, \partial F_1/\partial z = \partial F_3/\partial x,\) and \(\partial F_2/\partial z = \partial F_3/\partial y.\) There are many ways to find a function \(\varphi(x, y, z)\) such that \(\nabla \varphi = \mathbf{F},\) which is what we need to find. Here is one method. We will take antiderivatives of \(F_1\) with respect to \(x, F_2\) with respect to \(y,\) and \(F_3\) with respect to \(z\) respectively and then compare the results.

\[
\varphi(x, y, z) = \int (2xy + z^2)dx = x^2y + xz^2 + C_1(y, z)
\]
\[
\varphi(x, y, z) = \int (x^2 + 2yz)dy = x^2y + y^2z + C_2(x, z)
\]
\[
\varphi(x, y, z) = \int (y^2 + 2xz)dz = y^2z + xz^2 + C_3(x, y)
\]
It is very important that $C_1(y, z)$ is function of $y$ and $z$ and not just a constant, since we are “undoing” a partial derivative where we considered $y$ and $z$ as constants (similarly for $C_2(x, z)$ and $C_3(x, y)$). If we examine the three versions of $\varphi(x, y, z)$ we see that each version has at least one term in common. Therefore, we might try $\varphi(x, y, z) = x^2y + y^2z + xz^2$, which turns out to work in this case.

(b) $F = (\ln(xy))i + (\frac{xy}{y})j + (y)k$.

Solution. Note that $F$ is only defined for $x, y > 0$ or $x, y < 0$ and $F_1 = \ln(xy)$, $F_2 = x/y$, and $F_3 = y$ have continuous partials in these regions of the plane. Further, if $F = \nabla \varphi$, and hence $F$ is conservative, then the mixed second partials of $\varphi$ must be equal. But since $\partial F_2/\partial z = 0$ and $\partial F_3/\partial y = 1$, no such $\varphi$ could exist with $\nabla \varphi = (\ln(xy))i + (\frac{xy}{y})j + (y)k$.

c) $F = (e^x \cos y)i + (-e^x \sin y)j + (2z)k$.

Solution. By inspection, it is easy to see that $\varphi(x, y, z) = z^2 + e^x \cos y$ is a potential function for $F$. Otherwise, one could use a method similar to (a).