Example 1: Determine whether the following limits exist. If yes, find the limit. If not, justify.

(i) \( \lim_{(x,y) \to (0,0)} \arctan \left( \frac{1}{x^2 + y^2} \right) \),

(ii) \( \lim_{(x,y) \to (0,0)} \frac{x^4 - y^4}{x^4 + x^2 y^2 + y^4} \),

(iii) \( \lim_{(x,y,z) \to (0,0,0)} \sin \left( \frac{1}{x^2 + y^2 + z^2} \right) \).

Solution. (i) Convert to polar coordinates; i.e., set \( r = \sqrt{x^2 + y^2} \). Then \( r \to 0 \) as \((x, y) \to 0\). Therefore,

\[
\lim_{(x,y) \to (0,0)} \arctan \left( \frac{1}{x^2 + y^2} \right) = \lim_{r \to 0} \arctan \left( \frac{1}{r^2} \right) = \lim_{z \to -\infty} \arctan z = -\frac{\pi}{2},
\]

where at the last but one step we have substituted \( z = -1/r^2 \).

(ii) The substitution \( y = mx \) yields

\[
\lim_{(x,y) \to (0,0)} \frac{x^4 - y^4}{x^4 + x^2 y^2 + y^4} = \lim_{x \to 0} \frac{x^4(1 - m^4)}{x^4(1 + m^2 + m^4)} = \frac{1 - m^4}{1 + m^2 + m^4}.
\]

Hence if \((x, y) \to (0, 0)\) along the line \( y = 0 \) (where \( m = 0 \), then the limit is 1, whereas if \((x, y) \to (0, 0)\) along the line \( y = x \) (when \( m = 1 \)), the limit is 0. Therefore the limit does not exist.

(iii) Using spherical coordinates yields

\[
\lim_{(x,y,z) \to (0,0,0)} \sin \left( \frac{1}{x^2 + y^2 + z^2} \right) = \lim_{\rho \to 0} \sin \left( \frac{1}{\rho^2} \right) = \lim_{t \to 0^+} \sin \left( \frac{1}{t} \right).
\]

Since the graph of \( \sin(1/t) \) oscillates arbitrarily fast between -1 and 1 near \( t = 0 \), the limit does not exist.

Example 2: Determine the largest set of points in the \( xy \)-plane on which \( f(x, y) = \tan(1/(x + y)) \) defines a continuous function.

Solution. Because the tangent function is continuous on the set \( \mathbb{R} \setminus \{ \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2 \cdots \} \), the given function \( f \) has discontinuities whenever

\[
x + y = 0 \quad \text{or} \quad \frac{1}{x + y} = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \cdots.
\]
The set of discontinous points is therefore the union of an infinite number of parallel straight lines, given by
\[ x + y = \pm \frac{2}{(2n + 1)\pi}, \quad n = 0, 1, 2, \ldots \text{ and } x + y = 0. \]

Example 3: Compute the first-order partial derivatives of the following function:
\[ f(r, s, t) = (1 - r^2 - s^2 - t^2)e^{-rst}. \]
Solution.
\[
\frac{\partial f}{\partial r} = -2re^{-rst} - st(1 - r^2 - s^2 - t^2)e^{-rst} = e^{-rst}(r^2st + s^3t + st^3 - 2r - st),
\]
\[
\frac{\partial f}{\partial s} = -2se^{-rst} - rt(1 - r^2 - s^2 - t^2)e^{-rst} = e^{-rst}(rs^2t + r^3t + rt^3 - 2s - rt),
\]
\[
\frac{\partial f}{\partial t} = -2te^{-rst} - rs(1 - r^2 - s^2 - t^2)e^{-rst} = e^{-rst}(rst^2 + r^3s + rs^3 - 2t - rs).
\]

Example 4: Describe the level surface of the function \( f(x, y, z) = z + \sqrt{x^2 + y^2} \).
Solution. The level surface of \( f \) is defined by the equation \( f(x, y, z) = k \), where \( k \) is a constant. This translates to \( k - z = \sqrt{x^2 + y^2} \). The level surfaces of \( f \) are therefore the lower nappes of circular cones with vertices on the \( z \)-axis.

Example 5: Discuss the continuity of the function
\[ f(x, y) = \begin{cases} \frac{\sin(xy)}{xy} & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0. \end{cases} \]
Solution. The ratio of two continuous functions is always continuous, as long as the denominator does not vanish. Therefore \( f \) is continuous at every \((a, b)\) such that \( ab \neq 0 \). We therefore only need to verify continuity at a point where \( ab = 0 \). Using the substitution \( z = xy \) and the basic trigonometric limit \( \sin t/t \to 1 \) as \( t \to 0 \), we get
\[
\lim_{(x, y) \to (a, b)} \frac{\sin(xy)}{xy} = \lim_{z \to ab} \frac{\sin z}{z} = 1 = f(a, b).
\]
Therefore \( f \) is continuous at all \((a, b) \in \mathbb{R}^2\).

**Example 6**: Use implicit differentiation to find \( \partial z/\partial x \) and \( \partial z/\partial y \).

\[ yz = \ln(x + z). \]

*Solution.* Differentiating the equation with respect to \( x \) we get,

\[
\frac{y}{\partial x} = \frac{1 + \frac{\partial z}{\partial x}}{x + z}.
\]

Solving for \( \partial z/\partial x \) gives

\[
\frac{\partial z}{\partial x} = \frac{1}{y(x + z) - 1}.
\]

Similarly differentiation with respect to \( y \) yields

\[
z + y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial y}x + z,
\]

from which we obtain

\[
\frac{\partial z}{\partial y} = \frac{z(x + z)}{1 - y(x + z)}.
\]