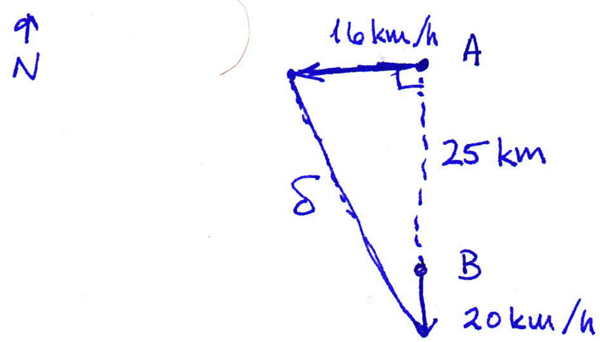
Q1 [10 marks]

At 1:00 p.m. ship A is 25 km due north of ship B. If ship A is sailing west at a rate of 16 km/h and ship B is sailing south at 20 km/h, find the rate at which the distance between the two ships is changing at 1:30 p.m. (Be sure to draw a diagram.)



Let $\delta(t)$ = distance between ship A and ship B at time t, t measured in hours.

$y(t)$ = distance of ship B from its 1 pm position.

$x(t)$ = distance of ship A from its 1 pm position.

$x(t) = 16t$, $y(t) = 20t$ & $\frac{dx}{dt} = 16$, $\frac{dy}{dt} = 20$.

$$\delta^2 = x^2 + (y+25)^2$$

$$\Rightarrow \cancel{2}\delta \frac{d\delta}{dt} = \cancel{2}x\frac{dx}{dt} + \cancel{2}(y+25)\frac{dy}{dt}$$

At 1:30, $t = 0.5$ so $x(0.5) = 16(1/2) = 8$

$y(0.5) = 20(1/2) = 10$.

$$\delta = \sqrt{8^2 + 35^2} = \sqrt{1289}$$

$$\Rightarrow \sqrt{1289}\cdot\frac{d\delta}{dt} = 8\cdot 16 + 35\cdot 20$$

$$\Rightarrow \frac{d\delta}{dt} = \frac{1}{\sqrt{1289}}(828) = \boxed{\frac{828}{\sqrt{1289}}\ \text{km/h}} \quad (\approx 23\ \text{km/h})$$

Q2 [10 marks]

A used Cesna 172 Skyhawk aircraft is purchased for \$56,000. The buyer predicts it will decline continuously in value at a rate of 4% per year.

(a) Write down a function to model the value of the aircraft t years from now.

$$A(t) = Pe^{-0.04t} = 56\,000e^{-0.04t}.$$

(b) What is the predicted value of the plane 5 years from now?

$$A(5) = 56{,}000\,e^{-0.04(5)} = \boxed{56000e^{-0.2}}$$

(c) In 5 years time, the buyer is forced to sell the plane for \$30,000. What constant annual rate of depreciation would have the buyer's \$56,000 aircraft worth only \$30,000 after 5 years? Assume again continuous depreciation.

Want r so that

$$30\,000 = 56\,000e^{-r\cdot 5}$$

$$\Rightarrow \quad e^{-r\cdot 5} = \frac{30\,\cancel{000}}{56\,\cancel{000}} = \frac{15}{28}$$

$$\Rightarrow \quad -5r = \ln\left(\frac{15}{28}\right)$$

(about 12.5% depreciation rate)

$$\Rightarrow \quad \boxed{r = -\frac{1}{5}\ln\left(\frac{15}{28}\right)}$$

(which is a pos. # since $\frac{15}{28} < 1$)

Q3 [10 marks]

Suppose that Lindo Cafe. sells 400 half-kilogram bags of Colombian coffee per week when it is priced at \$10 per 500 grams. For every \$1 per bag increase in price, it sells 10 fewer bags of coffee. Recall that the price elasticity of demand is given by $\epsilon(p) = \frac{p}{q}\frac{dq}{dp}$.

(*a*) Find the demand equation for Lindo's Colombian coffee. Use p for price and q for the demand.

$$q - 400 = -10(p - 10)$$

$$\Rightarrow \boxed{q = 400 - 10(p - 10)}$$

$q = 500 - 10p$ in simplest form.

(*b*) Compute $\epsilon(p)$ for this demand function.

$$\epsilon(p) = \frac{p}{q}\frac{dq}{dp} = \frac{p}{400 - 10(p-10)} \cdot (-10)$$

$$\boxed{\epsilon(p) = \frac{-10p}{500 - 10p}} = \frac{-p}{50 - p}$$

(*c*) If the price is \$12 and increases by 4%, what is the percentage change in demand? (Hint: Use the price elasticity of demand to answer this question.) You may leave your answer in the simplest calculator-ready form you can find.

$$\epsilon(12) = \frac{-10(12)}{500-10(12)} = -\frac{120}{380} = -\frac{6}{19}$$

Recall that $\epsilon(p) = \dfrac{\%\text{ change in demand}}{\%\text{ change in price}}.$

$$\Rightarrow \%\text{ change in demand} = \epsilon(p)\cdot\%\text{ change in price}$$
$$= -\frac{6}{19} \times 0.04 \times 100\% = \boxed{\frac{24}{19}\%} \approx 1.26\%$$

(*d*) Will the Lindo Cafe's revenue increase or decrease as a result of the price change in part (c)? Explain your answer.

Note that $\epsilon(12) = -\dfrac{6}{19}$

so $|\epsilon(12)| = \dfrac{6}{19} < 1$. inelastic

Thus, a small increase in price from \$12 results in an $\boxed{\text{increase}}$ in revenue.

Q4 [5 marks]

For the function

$$f(x) = \frac{x^2-1}{x^2-4}$$

determine all of the following if they are present: (i) critical points (where $f'(x) = 0$ or $f'(x)$ does not exist), local maxima and minima, intervals where $f(x)$ is increasing or decreasing; (ii) inflection points and intervals where $f(x)$ is concave upward or downward; (iii) asymptotes (horizontal, vertical, slant). Sketch the graph of $y = f(x)$, giving the (x, y) coordinates for all of the points of interest above. **Please make your sketch big enough to see clearly all features of interest.**

You may use, without demonstrating it, the fact that $f''(x) = \dfrac{6(3x^2+4)}{(x^2-4)^3}$.

$$f'(x) = \frac{2x(x^2-4)-(x^2-1)(2x)}{(x^2-4)^2} = \frac{-6x}{(x^2-4)^2} = 0$$

$x \neq \pm 2$

$\Rightarrow$ CP: $x=0$ & $x=\pm 2$ are ~~all~~. important points of consideration.

	$x<-2$	$x=-2$	$-2<x<0$	$x=0$	$0<x<2$	$x=2$	$x>2$
f'	$+$		$+$	0	$-$		$-$
f	$\nearrow$	vertical asymptote	$\nearrow$	local max.	$\searrow$	vertical asymptote	$\searrow$

Note: $\displaystyle\lim_{x\to\pm\infty} \frac{x^2-1}{x^2-4} = \lim_{x\to\pm\infty} \frac{\cancel{x^2}(1-1/x^2)}{\cancel{x^2}(1-4/x^2)} = 1$

So $y=1$ is a horizontal asymptote.

(QUESTION 4 CONTINUED)

Now, $f''(x) = \dfrac{6(3x^2+4)}{(x^2-4)^2}$ is never 0, but $x = \pm 2$ are still points to consider.

		$x=-2$		$x=2$	
f''	$+$		$-$		$+$
f	$\cup$		$\cap$		$\cup$

$x = \pm 2$ ARE <u>NOT</u> inflection pts.