

## Some Notes on Revenue, Cost, and Profit

compiled by Patrick Chan

Summer 2011

### Marginal Cost:

Common Economics Definition: The additional cost of producing 1 extra unit.

Formal Definition: The derivative of cost with respect to quantity ( $\frac{dC}{dq}$ ).

EXAMPLE  $C(q) = 2000 + 10q^2$

$$\text{Marginal Cost} = \frac{dC}{dq} = 20q$$

### Fixed Costs:

Definition: Costs of production that do not change with production (Quantity).

EXAMPLE  $C(q) = 2000 + 10q^2$

$$\text{Fixed Costs} = 2000$$

### Variable Costs: (Note: **Not the Same as MARGINAL COSTS!**)

Definition: Costs that vary with quantity (anything with the term  $q$  in the cost function).

EXAMPLE  $C(q) = 2000 + 10q^2$

$$\text{Variable Cost} = 10q^2$$

### Marginal Revenue

Common Economics Definition: Additional Revenue for 1 extra unit of production.

Formal Definition: Marginal Revenue =  $\frac{dR}{dq}$ .

We know that revenue ( $R$ ) is computed as Price x Quantity ( $p * q$ ):

$$\boxed{R = pq.}$$

EXAMPLE Let  $p = 30 - 5q$  be the demand equation.  
Since we know that  $R = pq$

$$R = 30q - 5q^2. \quad (1)$$

The derivative of the revenue function with respect to quantity will be

$$\frac{dR}{dq} = 30 - 10q. \quad (2)$$

### Profit

Quite simply, Profit = Revenue - Cost.

We will use  $P$  to represent Profit so be careful not confuse it with  $p$ , which is Price. This gives us the basic equation

$$P = R - C. \quad (3)$$

Profit is maximized at the level of output where marginal profit is zero. In economics, this is usually taken to mean that an additional unit of output  $q$  will not change the profit. In this course, we will use the derivative definition of marginal and so profit is maximized when

$$\frac{dP}{dq} = 0. \quad (4)$$

Taking the derivative on both sides, we obtain

$$\frac{dP}{dq} = \frac{dR}{dq} - \frac{dC}{dq} = 0, \quad (5)$$

which means

$$MR - MC = 0. \quad (6)$$

That is, we will examine output values  $q$  at which marginal revenue equals marginal cost:

$$\boxed{MR = MC.} \quad (7)$$

This means we will solve for  $q$  after setting  $MR = MC$

EXAMPLE

Find the optimal output in order to maximize profit given:

$$\frac{dC}{dq} = 8q, \quad (8)$$

$$\frac{dR}{dq} = 900 - 10q. \quad (9)$$

SOLUTION

$$\text{Set } \frac{dC}{dq} = \frac{dR}{dq}:$$

$$8q = 900 - 10q. \quad (10)$$

Now solve for  $q$ .

$$18q = 900, \quad (11)$$

so

$$\boxed{q = 50.} \quad (12)$$

How do you know this is a *maximum*? (The answer lies either in some high school algebra or in an argument using calculus.)

EXAMPLE

Find the optimal output in order to maximize profit given that revenue function is  $R(q) = 9000Q - 207q^2$  and the cost function is  $C(q) = 18q^2$ .

SOLUTION

We need to find  $MR$  and  $MC$  first.

$$\frac{dC}{dq} = 36q \quad (13)$$

and

$$\frac{dR}{dq} = 9000 - 414q. \quad (14)$$

Equating (13) and (14), we get

$$36q = 9000 - 414q, \quad (15)$$

which gives

$$\boxed{q = 20}. \quad (16)$$

### Demand Curve

Definition: The *demand curve* is a curve or schedule (table of values) showing the total quantity of a good of uniform quality that buyers want to buy at each price during a particular period of time provided that all other things are held constant. Mathematically, this means it is a relation between the price  $p$  and the quantity demanded  $q$ .

Linear demand means this relation is a straight line; *viz.*

$$q = a - bp, \quad (17)$$

where  $q$  is the quantity demanded,  $p$  is the demand price, and  $a$  and  $b$  are given parameters (constants).

The so-called Law of Demand says that any real demand curve is always downwards sloping when the vertical axis is price  $p$  and the horizontal axis is demand quantity  $q$ . This means that if price is increased, demand will fall. Note that mathematically, this labelling of the axes is arbitrary (though seems to be a matter for strong opinions amongst some!).

#### EXAMPLE

Suppose the demand curve for a product produced by a firm is given by  $q = 1350 - 5p$  and the cost function is  $C(q) = 60q + 4q^2$ . Find the profit maximizing output for the firm.

#### SOLUTION

We will need to find the marginal revenue  $MR$  using the given demand function.

Recall that  $R = pq$ . If we isolate for  $p$  in the demand equation, we can multiply everything by  $q$  to find the revenue function:

$$5p = 1350 - q, \quad (18)$$

which yields

$$p = 270 - \frac{q}{5}. \quad (19)$$

Now,

$$R = pq \quad (20)$$

and so, as a function of  $q$ ,

$$R(q) = 270q - \frac{q^2}{5}. \quad (21)$$

This means that marginal revenue is

$$\frac{dR}{dq} = 270 - \frac{2q}{5}. \quad (22)$$

The marginal cost  $\frac{dC}{dq}$  is

$$\frac{dC}{dq} = 60 + 8q. \quad (23)$$

Setting  $\frac{dC}{dq} = \frac{dR}{dq}$ , we get

$$270 - 0.4q = 60 + 8q, \quad (24)$$

which gives

$$q = 25. \quad (25)$$

How do we know this value of  $q$  gives us a *maximum* profit?

We are interested often in the *break-even* points. That is, these are the points at which the revenues balance the costs and we make no profit. We want to understand these points since they help us decide our production behaviour since varying production when we are close to a break-even point can mean the difference between profitability and generating a loss.

Mathematically, to find the break even points, we just set  $R(q) = C(q)$  and solve for  $q$ . In this example, this gives

$$270q - \frac{q^2}{5} = 60q + 4q^2. \quad (26)$$

Solving gives us

$$4.2q^2 = 210q, \quad (27)$$

or

$$\boxed{q = 0 \text{ or } 50.} \quad (28)$$

Note that  $q = 0$  would mean the firm produces nothing, and therefore has no revenue and no cost (note that our cost function does not include fixed costs, which is a tad unrealistic). Hence, the more interesting break-even point is to produce  $q = 50$ . What happens if we increase  $q$  from this value? If we decrease it?