The University of British Columbia

9 October 2013

Alternate Midterm for All Sections of MATH 104 and 184

Closed book examination Time: 60 minutes

Last Name Solutions **First** ______________

Signature ______________

Student Number ______________

MATH 104 or MATH 184 (Circle one) **Section Number:**______________

Special Instructions:

No memory aids are allowed. No calculators. No communication devices. Show all your work; little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page. **Where boxes are provided for answers, put your final answers in them.**

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		18
2		8
3		10
4		7
5		7
Total		50

[21] 1. **Short Answer Questions**. Each question is worth 3 points. Put your final answer in the box provided, but NO CREDIT will be given for the answer without the correct accompanying work.

(a) Evaluate $\lim_{x\to 2} \dfrac{3x^2 - 7x + 2}{2 - x}$.

$$\lim_{x\to 2} \frac{(3x-1)(x-2)}{2-x}$$ 1 mark

$$= \lim_{x\to 2} -(3x-1) = -5.$$

1 mark

Answer: -5

3

(b) Evaluate $\lim_{y\to 3} \dfrac{-5y}{\sqrt{4y-3}}$.

This function is continuous at $y=3$, so 1 mark

$$\lim_{y\to 3} \frac{-5y}{\sqrt{4y-3}} = \frac{-5(3)}{\sqrt{4(3)-3}} = \frac{-15}{3} = -5$$

1 mark

Answer: -5

3

(c) Find the derivative of $f(x) = \dfrac{\sin(x^3)}{x^2}$.

Answer:

$$f'(x) = \frac{3x^2 \cos(x^3)\cdot x^2 - \sin(x^3)\cdot 2x}{x^4}$$ 3

1 mark knowing quotient rule

1 mark knowing chain rule

no need to simplify

(d) Find $g'(z)$ if $g(z) = z^5(e^z + 1)$.

Answer:

$$g'(z) = 5z^4(e^z+1) + z^5 e^z$$

3

1 mark for knowing product rule

1 mark for correctly taking a derivative of a piece of this.

(e) Let $f(x) = x\cos(x)$. Find the equation of the tangent line to $f(x)$ at $x = \pi/4$.

$f(\pi/4) = \frac{\pi}{4}\cdot\frac{1}{\sqrt{2}}$.

Answer:

1 mark for product rule

$$f'(x) = \cos(x) - x\sin(x)$$

$$f'(\pi/4) = \cos(\pi/4) - \frac{\pi}{4}\sin(\pi/4)$$

← note! we accept this calculator ready form.

$$= \frac{1}{\sqrt{2}} - \frac{\pi}{4}\cdot\frac{1}{\sqrt{2}}.$$

1 mark for correct slope

Tangent line

$$\boxed{y - \frac{\pi}{4\sqrt{2}} = \left(\frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}}\right)\left(x - \pi/4\right)}$$

3

1 mark for correct form of a tangent line

(f) Find the y-intercept of the tangent line to $y = f(x) = \dfrac{x+5}{x-1}$ at the point on this curve corresponding to $x = 3$.

Answer: $17/2$

3

$$f'(x) = \frac{1(x-1) - (x+5)1}{(x-1)^2} = \frac{-6}{(x-1)^2}$$

1 mark for derivative

$$f'(3) = \frac{-6}{(3-1)^2} = -\frac{3}{2}$$

At $x=3$, $f(3) = \frac{8}{2} = 4$

Tangent line

$$y - 4 = -\frac{3}{2}(x-3)$$

1 mark for eqn of tangent line

$$\Rightarrow y = -\frac{3}{2}x + \underbrace{17/2}_{y\text{-intercept}}$$

Alternate: $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

[8] **2. Definition of the Derivative.**

2 marks

(a) [3] Carefully state the definition of the derivative of a function $f(x)$ at a point $x = a$.

A function f has a derivative at $x=a$ if $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists. 1 mark

(b) [5] Use the definition of the derivative to compute the derivative of $f(x) = \frac{2}{x-1}$ at $x = 5$. NO CREDIT will be given for using any other method.

$$f'(5) = \lim_{x \to 5} \frac{f(x) - f(5)}{x - 5}$$ 1 mark

$$= \lim_{x \to 5} \frac{\frac{2}{x-1} - \frac{2}{5-1}}{x-5}$$ 1 mark

$$= \lim_{x \to 5} \frac{\frac{4 - (x-1)}{2(x-1)}}{x-5}$$ 2 marks algebra

$$= \lim_{x \to 5} \frac{\frac{\cancel{5-x}^{\,-1}}{2(x-1)}}{\cancel{x-5}} = \boxed{-\frac{1}{8}}$$

1 mark final answer

[10] **3**. Math Cycles Co. makes and sells the world's first bicycle made of recycled plastic bottles! When each bicycle is sold for \$500, the weekly demand is 4,000 units. For every \$1 increase in the price of each unit, the number of bicycles sold per week decreases by 10. Assume that it costs \$300 to produce each bicycle.

(a) [2] Find the linear demand equation for the Math Cycles bicycle. Use p for the unit price and q for the weekly demand.

Given $(q_0, p_0) = (4000, 500)$ slope $= \frac{\Delta p}{\Delta q} = \frac{1}{-10} = m$

1 mark for slope

$$p - p_0 = m(q - q_0)$$

$$\Rightarrow p - 500 = -\frac{1}{10}(q - 4000) \Rightarrow \boxed{p = -\frac{1}{10}q + 900}$$

1 mark for equ.

(b) [1] Find the weekly cost function $C(q)$ *as a function of* q.

$$\boxed{C(q) = 300q}$$

1 mark

(c) [1] Find the weekly revenue function $R(q)$ *as a function of* q.

$$R(q) = p(q) \cdot q = \left(-\frac{1}{10}q + 900\right)q$$

$$\Rightarrow \boxed{R(q) = -\frac{1}{10}q^2 + 900q}$$

1 mark

(d) [2] Find the break-even points for the Math Cycles bicycle. Give both the price p and quantity q at each of these points.

Break-even: $R(q) = C(q)$

$\Rightarrow -\frac{1}{10}q^2 + 900q = 300q$ 1 mark

$\Rightarrow -\frac{1}{10}q^2 + 600q = 0 \Rightarrow \boxed{q = 0 \text{ or } q = 6000}$

1 mark

(e) [2] Find the marginal profit function $MP(q)$.

$P(q) = R(q) - C(q) = -\frac{1}{10}q^2 + 900q - 300q$

$= -\frac{1}{10}q^2 + 600q$ 1 mark

$\boxed{MP(q) = P'(q) = -\frac{q}{5} + 600}$ 1 mark

(f) [2] Suppose that Math Cycles is producing and selling $\hat{q}$ bicycles, where $\hat{q}$ corresponds to the largest q-value of all the break-even points. Should Math Cycles increase or decrease the price of its bicycles to increase its profit? Explain your answer.

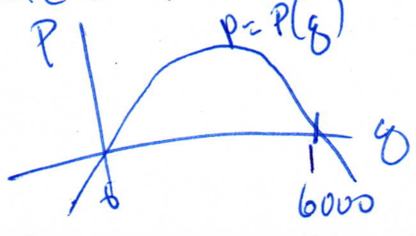
$\hat{q} = 6000.$ $MP(\hat{q}) = MP(6000) = -\frac{6000}{5} + 600$

$= -600 < 0$ 1 mark

Since $MP(\hat{q}) < 0$, Math Cycles needs to decrease demand to increase Profit. 1 mark.

$\Rightarrow$ Increase price P.

A graphical solution is acceptable



[7] 4. Let a and b denote constants. Find the values of a and b so that the function

$$f(x) = \begin{cases} 2 + 3e^x & \text{if } x < 0, \\ ax + b & \text{if } x \geq 0 \end{cases}$$

is differentiable everywhere.

(1) Each piece of f is differentiable on its part of the domain away from x=0. 1 mark

(2) Diff ⇒ continuous. 1 mark

(3) To be diff, f must be continuous at x=0:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = b$$

1 mark for looking at limits at x=0

$$\Rightarrow \lim_{x\to 0^-} 2+3e^x = \boxed{5 = b}$$

1 mark for b

(4) Need to match derivatives at x=0 as well:

$$\left.\frac{d}{dx}(2+3e^x)\right|_{x=0^-} = 3$$

2 marks

$$\left.\frac{d}{dx}(ax+5)\right|_{x=0^+} = a$$

$$\Rightarrow \boxed{a=3}$$

1 mark

[7] **5** Does $f(x) = \cos^2(x) - \ln(x)$ have a root in $(1, \pi/2)$? Explain your answer carefully, and be sure to justify each assertion you make.

① f is continuous on $(0, \infty)$

~~so is~~

1 mark for stating continuity

② We can apply the Intermediate Value Theorem to $f(x)$ on $[1, \pi/2]$.

2 marks

③ $f(1) = \cos^2(1) - \ln(1) = \cos^2(1) > 0$

1 mark

since $\cos(1) \neq 0$.

1 mark explain why $\cos^2(1) > 0$.

④ $f(\pi/2) = \cos^2(\pi/2) - \ln(\pi/2)$

$= -\ln(\pi/2) < 0$

since $\ln(\pi/2) > 0$

since $\pi/2 > 1$.

1 mark explain why $f(\pi/2) < 0$

⑤ So there is a root of f in $(1, \pi/2)$.

1 mark.