The University of British Columbia

8 November 2017

Midterm for MATH 104

Closed book examination Time: 50 minutes

Family Name: _____ **Given Name:** _____

Signature: _____

Student Number: _____

Section Number:_____

Special Instructions:

No memory aids are allowed. No calculators. No communication or other electronic devices. Show all your work; little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page. **Where boxes are provided for answers, put your final answers in them.**

Midterms written in pencil will not be considered for regrading.

Rules governing examinations

• Each candidate must be prepared to produce, upon request, a UBCcard for identification.

• Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

• Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.

• Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

• Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		20
2		10
3		10
4		10
Total		50

[20] **1. Short Answer Questions**. Each question is worth 3 points except part (g), which is worth 2 points. Put your final answer in the box provided, but NO CREDIT will be given for the answer without the correct accompanying work.

(a) If $y = xy + x^2 + 1$, find the equation of the tangent line to this curve at the point $(-1, 1)$.

Answer:
$x + 2y = 1$

Soln: Differentiating the given equation w.r.t x, we get

$$y' = xy' + y + 2x$$

$$y'(1 - x) = y + 2x$$

$$y' = \frac{y + 2x}{1 - x}$$

At the point $(-1, 1)$,

$$y' = -\frac{1}{2}$$

Equation of the tangent line at the point $(-1, 1)$, is given by

$$y - 1 = -\frac{1}{2}(x + 1)$$

$$x + 2y = 1$$

(b) Find $\dfrac{dy}{dx}$ where $x^2e^2 + 2^y = 5$.

Answer:
$-\dfrac{xe^2}{2^{y-1}\log 2}$

Soln: Differentiating the given equation w.r.t x, we get

$$2xe^2 + \log 2 \cdot 2^y \cdot y' = 0$$

$$y' = \frac{-2xe^2}{2^y \log 2} = -\frac{xe^2}{2^{y-1}\log 2}$$

(c) Find the derivative of $f(x) = x^{\log x}$.

Answer:
$2x^{\log x - 1} \cdot \log x$

Soln: Taking 'log' on both sides of the given equation, we get

$$\log f(x) = \log x^{\log x}$$

$$\log f(x) = \log x \cdot \log x = (\log x)^2$$

Differentiating the above equation w.r.t x, we get

$$\frac{f'(x)}{f(x)} = 2\log x \cdot \frac{1}{x}$$

$$f'(x) = f(x) \cdot \frac{2\log x}{x}$$

$$f'(x) = x^{\log x} \cdot \frac{2\log x}{x} = 2x^{\log x - 1} \cdot \log x$$

(d) What is the annual rate of interest that would need to be compounded continuously to yield \$10000 after 5 years if one were to start with \$2500?

Answer:
$\frac{\log 4}{5}$

Soln: For continuously compounded interest,

$$A = Pe^{rt}$$

Here A=\$10,000, P=\$2500, t = 5 yrs, so

$$10000 = 2500e^{r \cdot 5}$$

$$e^{5r} = \frac{10000}{2500} = 4$$

$$5r = \log 4$$

$$r = \frac{\log 4}{5}$$

(e) Find the x-coordinate of the absolute maximum of $f(x) = x\sqrt{2 - x^2}$ on its domain.

Answer:
$x = 1$

Soln: For the given function $f(x) = x\sqrt{2 - x^2}$ to exist:

$$2 - x^2 \geq 0$$

So domain of this function is given by:

$$x \in [-\sqrt{2}, \sqrt{2}]$$

Differentiating the function w.r.t x, we get

$$f'(x) = 1 \cdot \sqrt{2 - x^2} + x \cdot \frac{1}{2\sqrt{2 - x^2}} \cdot -2x$$

$$f'(x) = \frac{2 - 2x^2}{\sqrt{2 - x^2}}$$

On the interval $[-\sqrt{2}, \sqrt{2}]$, the critical points are given by:

$$f'(x) = 0 \implies x = \pm 1$$

Comparing the function values at the end points $x = \pm\sqrt{2}$ and the critical points $x = \pm 1$,

$$f(-\sqrt{2}) = 0, f(-1) = -1, f(1) = 1, f(\sqrt{2}) = 0$$

The absolute maximum $f(1) = 1$ is at the point $x = 1$.

(f) If the base b of a triangle is increasing at a rate of 3 centimetres per second while its height h is decreasing at a rate of 3 centimetres per second, which of the following must be true about the area A of the triangle?

(A) A is always increasing.

(B) A is always decreasing.

(C) A is decreasing only when $b < h$.

(D) A is decreasing only when $b > h$.

(E) A remains constant.

Answer:
D

Soln: Area of a triangle with base b and height h is given by:

$$A = \frac{1}{2} \cdot b \cdot h$$

Differentiating the above equation w.r.t time variable t, we get

$$\frac{dA}{dt} = \frac{1}{2}\left[b \cdot \frac{dh}{dt} + \frac{db}{dt} \cdot h\right]$$

Giiven $\frac{db}{dt} = 3\ cm/s$ and $\frac{dh}{dt} = -3\ cm/s$, we have

$$\frac{dA}{dt} = \frac{3}{2}(h - b)$$

By First Derivative test, when $\frac{dA}{dt} < 0, A$ is decreasing.

So for $h < b, \frac{dA}{dt} < 0 \implies A$ is decreasing.

(g) Two runners start a race at the same moment and finish in a tie (i.e., at the same time). Which of the following statements must be true?

(A) The runners' speeds at the end of the race must have been exactly the same.

(B) At some point during the race one runner was leading the other runner.

(C) The runners must have had the same speed at exactly the same time at some point in the race.

(D) The runners had to have the same speed at some moment during the race, but not necessarily at the same time.

Answer:
C

Soln: We define $P_1(t)$ and $P_2(t)$ as the position function for the two runners on the interval $[0, T]$, T being the time at which they reach the finish line. Assuming both the runners reached the finish line without stopping anywhere, we can say that the position functions are continuous and differentiable in the interval $(0, T)$.

Define a function: $S(t) = P_1(t) - P_2(t)$
Then

$$S(0) = S(T) = 0$$

and

$$S'(T) = P_1'(T) - P_2'(T)$$

By Rolle's theorem, there exists a point $\tau \in (0, T)$ such that

$$S'(\tau) = 0 \implies P_1'(\tau) - P_2'(\tau) = 0$$

At time $t = \tau$,

$$P_1'(\tau) = P_2'(\tau),$$

i.e., the runners must have had same speed at exactly the same time at some point in the race.

[10] **2**. The demand relationship for Poe and Company's new fountain pen is

$$qp + 5p + 2q = 305,$$

where p is the price per unit and q is the number of fountain pen's per day at price p. The price elasticity of demand is given by $E = \frac{p}{q}\frac{dq}{dp}$.

(a) Compute the elasticity of demand, $E(p)$, as a function of p.
Soln: For the given demand realtion, we can write

$$q(p) = \frac{305 - 5p}{p + 2}$$

Also, implicitly differenting the given demand relation w.r.t the price 'p', we get

$$\frac{dq}{dp}p + q + 5 + 2\frac{dq}{dp} = 0$$

$$\frac{dq}{dp} = -\frac{q + 5}{p + 2}$$

Using the earlier expression for 'q', we get

$$\frac{dq}{dp} = -\frac{315}{(p + 2)^2}$$

Elasticity of demand is given by

$$E = \frac{p}{q} \cdot \frac{dq}{dp}$$

$$E(p) = \frac{p(p + 2)}{(305 - 5p} \cdot -\frac{315}{(p + 2)^2}$$

$$E(p) = -\frac{315p}{(p + 2)(305 - 5p)}$$

(b) What is $E(p)$ when $p = \$5$?
Soln:

$$E(5) = -\frac{315 \times 5}{(5 + 2)(305 - 5 \times 5)} = -\frac{45}{56}$$

(c) If the price is lowered from \$5 by 2%, what is the percentage change in the demand for fountain pens?
Soln:

$$E = \frac{\%\Delta q}{\%\Delta p}$$

$$-\frac{45}{56} = \frac{\%\Delta q}{-0.02}$$

$$\%\Delta q = -\frac{45}{56} \cdot -0.02 = \frac{45}{28}\% \approx 1.6\%$$

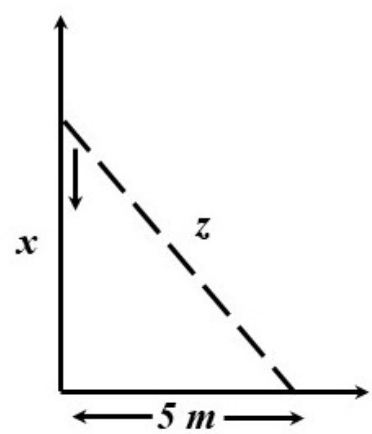
The demand for fountain pens increases by 1.6%.

(d) Does revenue increase or decrease when the price is lowered from \$5 by 2%? Explain your answer.
Soln: $|E| = \frac{45}{56} < 1 \implies$ the product is price inelastic.
When the price is lowered from \$5 by 2%, the % increase in demand is less than the % decrease in price, so the revenue decreases.

[10] **3**. A 20 m long extension ladder leaning against a wall starts collapsing in on itself at a rate of 2 $m\,s^{-1}$, while the foot of the ladder remains a constant 5 m from the wall. How fast is the ladder moving down the wall after 3.5 seconds?

Answer:
$-\frac{13}{6}\ m/s$

Soln:



By the Pythagoras theorem, we have

$$z^2 = x^2 + 5^2$$

$$x = \sqrt{z^2 - 25}$$

Given: $\frac{dz}{dt} = -2\ m/s$, also at $t = 0, z = 20\ m$
Unknown: $\frac{dx}{dt}$ at $t = 3.5\ s$??
Differentiating the equation relating 'x' and 'z' w.r.t the time variable 't', we get

$$\frac{dx}{dt} = \frac{1}{2\sqrt{z^2 - 25}} \cdot 2z \cdot \frac{dz}{dt}$$

$$\frac{dx}{dt} = \frac{z}{\sqrt{z^2 - 25}} \cdot \frac{dz}{dt}$$

At $t = 3.5s$, $\frac{dz}{dt} = -2\ m/s$, and $z = 20 + \frac{dz}{dt} \cdot (3.5) = 13\ m$
Hence, at $t = 3.5s$,

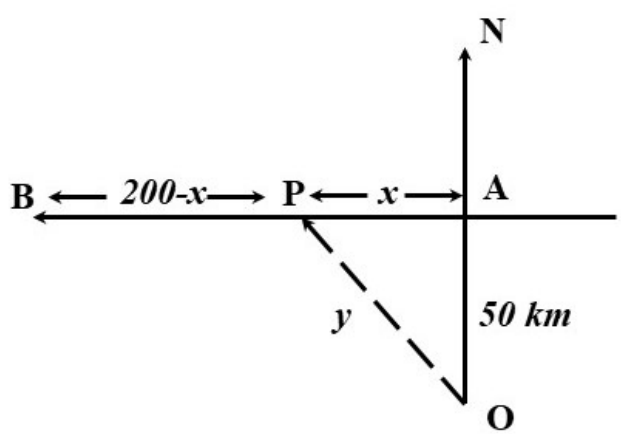
$$\frac{dx}{dt} = \frac{13 \cdot (-2)}{\sqrt{(13)^2 - 25}}$$

$$\frac{dx}{dt} = -\frac{13}{6}\ m/s$$

[10] **4.** A cargo ship sits at point O in the sea, 50 km due south of the nearest point A on the shore line that runs east-west. You want to get the cargo to a port B on the shore 200 km west of A. Suppose that it costs \$200 per kilometre to move the cargo by ship, and \$150 per kilometre to move the cargo along the shoreline in a truck. Toward what point on the shore should you head to minimize the cost in taking the cargo from O to B? (**Give your answer relative to the point A on the shore.**)

Answer:
$\sqrt{\frac{22500}{7}} \approx 56.69\ km$

Soln:



Let 'x' be the distance of the point P from the point A on the shore such that $x \in [0.200]$ and would minimize the cost C in taking the cargo from O to B such that the cost function is given by:

$$C = C_{sea} \cdot OP + C_{land} \cdot PB$$

$$C = C_{sea} \cdot y + C_{land} \cdot (200 - x)$$

By the Pythagoras theorem, we have

$$OP^2 = OA^2 + AP^2$$

$$y^2 = x^2 + 50^2$$
$$y = \sqrt{x^2 + 2500}$$

So the cost function as a function of 'x' can be written as:

$$C(x) = C_{sea} \cdot \sqrt{x^2 + 2500} + C_{land} \cdot (200 - x)$$

Given:
$C_{land} = \$150$ per km and $C_{sea} = \$200$ per km
Unknown:
Minimize the cost function $C(x)$?
Therefore,

$$C(x) = 200 \cdot \sqrt{x^2 + 2500} + 150 \cdot (200 - x)$$

In order to minimize 'C', we have to find the critical points using:

$$C'(x) = 0$$

$$\frac{2x}{\sqrt{x^2 + 2500}} \cdot 200 + (-1) \cdot 150 = 0$$
$$200x - 150\sqrt{x^2 + 2500} = 0$$
$$7x^2 = 9$$
$$x = \sqrt{\frac{22500}{7}} \approx 56.69\ km$$

Evaluating the second derivative of the cost function, we get:
$C''(x) = \dfrac{50000}{(x^2 + 2500)^{\frac{3}{2}}} > 0$ for all $x \in [0.200]$.

Thus, $x = 56.69$ is a point of minimum.