MATH 101: Week 9 and 10 Learning Goals
March 11, 2014

This week we begin our unit on Sequences and Series (Chapter 11 in Stewart). Our main goal of this entire unit (Weeks 9 to 12), is to get to Taylor series, which are the natural extension of the Taylor polynomials you studied in first term. In weeks 9 and 10, you will cover material from sections 11.1, 11.2, 11.3, 11.4, 11.5, 11.6, and 11.8. (The last is likely to continue into week 11).

Learning Goals

The specific learning goals for this week are that by the end of the week and review homework, you should be able to:

1. Conjecture a formula or a recurrence relation which defines all the terms of a **sequence**. Determine if a sequence is **convergent**, **divergent**, **increasing**, **decreasing** or **bounded**. Evaluate the **limit** of a convergent sequence.

2. Be able to use the **sigma notation** to represent a **series**, and recognize that the sum of the series is the limit of the partial sums. Find the sum of a **geometric series** or determine that a geometric series is divergent. Be able to recognize **telescoping sums** and **harmonic series**. Use the test for divergence to determine if a series can be concluded to be divergent.

3. Use the **integral test** to determine if a series is convergent or divergent. Use integrals to approximate the sum of a series to a desired accuracy.

4. Use the **comparison test** to determine if a series is convergent or divergent by proposing a different series whose convergence properties can be easily determined.

5. Be able to recognize **alternating series**. Use the alternating series test to determine whether a series can be concluded to be convergent. Use the **alternating series estimation theorem** to approximate the sum of an alternating series.

6. Be able to distinguish between **absolute convergence** and **conditional convergence** of a series. Use the **ratio test** to determine whether a series is absolutely convergent or divergent.

7. Be able to recognize functions that are represented as a **power series**. Determine the **radius of convergence** and the **interval of convergence** of a power series.

Potential Learning Approaches and Issues

1. The main goal of this part of the course is to get you to a reasonable understanding of Taylor series as representations of functions. You will recall from last term that we introduced you to Taylor Polynomials: Given information about a function and its derivatives at a particular point, $x = a$, we constructed an nth polynomial using these data, $P_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^2)$, where $a_n = f^{(n)}(a)/n!$. This gave us the ability to compute approximate values for functions such as e^x and $\ln(x)$, for example. In Chapter 11, we try to make sense of what happens if we keep adding more and more terms to produce a *series* $\sum_{n=1}^{\infty} a_n(x-a)^n$. What does this 'infinite sum" mean? How is it useful?

2. With this in mind, we note that the easiest thing for us to do is ask the question: suppose we had the polynomial $P_n(x)$ and added one more term, and the added another term, and so on. This generates a *sequence* of polynomials, or *partial sums*, and we can ask what happens to this sequence as n goes to infinity.

   Thus, section 11.1 to 11.6 provide us with some technical tools with which to tackle Taylor series. As you work through these ideas, if might be good to keep in mind our ultimate goal.

3. Indeed, consider two important Taylor series:

   $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

   and

   $$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots .$$

   The former gives us a way to compute e^x for *all* real numbers x, but the latter, as we shall see, only makes sense for $-1 < x \leq 1$. With these two simple examples, you can work through many of the key ideas in this chapter relevant to understanding Taylor series. For the exponential function, it is possible to have a Taylor series that *converges* everywhere, but for the logarithm function, we can only find a Taylor series that converges on a particular interval. You will learn how to determine what happens for any given Taylor series, but thinking about these two examples may be useful in keeping things concrete for you.

4. Students often find the notation a struggle to master, and it is worth spending time thinking carefully about it. I recommend that you write out a few terms of a series, for example, to see what the sum notation translates to as an actual sum of terms.

5. When thinking about *convergence* and *divergence* it is useful to compare to the ideas as we used them for improper integrals. (Indeed, section 11.3 makes a link between these ideas.)

6. Most of the time, you will find the Comparison Test of section 11.4 to be the most useful test. Indeed, it is worth building up enough basic examples of convergent and divergent series to be able to use it effectively. The p-series, $\sum_{n=1}^{\infty} 1/n^p$, and the particular case of the *harmonic series* $\sum_{n=1}^{\infty} 1/n$ are particularly useful.

7. It is useful to know the following relationships between the rates of growth of some basic sequences. By $\{a_n\} \ll \{b_n\}$ we mean that the sequence $\{b_n\}$ grows faster than the sequence $\{a_n\}$. Thus, $\{\ln n\} \ll \{n\} \ll \{n \ln n\} \ll \{n^2\} \ll \{e^n\} \ll \{n\} \ll \{n^n\}$.

8. In section11.1, ignore the formal definition of the limit of a sequence (Definition 2), and the formal definition of what it means for a sequence to go to infinity (Definition 5). We did not cover l'Hôpital's Rule in term 1, so ignore Example 6.

9. In sections 11.3 and 11.4, ignore estimating sums.

10. In section 11.6, ignore the Root Test.

Suggested Problems

Section 11.1: 1, 2, 7, 9, 13, 17, 27, 29, 39, 49, 79, 91* (this last one is an interesting problem)

Section 11.2: 1, 15, 17, 31, 36, 39, 45, 59, 77*, 87*

Section 11.3: 1, 2, 5, 7, 15, 21, 23, 25

Section 11.4: any (or all!) of the problems in 3 to 32.

Section 11.5: 1, and (or all!) of the problems between 5 and 20.

Section 11.6: 2, 3, 5, 8, 13, 25, 35.

Section 11.8: 3, 5, 7, 17, 20, 41.