

## MATH 101: Weeks 11 and 12 Learning Goals

March 26, 2014

We continue our unit on Sequences and Series (Chapter 11 in Stewart). Our main goal of this entire unit (Weeks 9 to 12), is to get to Taylor series, which are the natural extension of the Taylor polynomials you studied in first term. In weeks 10 and 11, you will cover material from sections 11.8, 11.9 and 11.10.

### Learning Goals

The specific learning goals for this week are that by the end of the week and review homework, you should be able to:

1. recognize functions that are represented as a **power series**;
2. determine the **radius of convergence** and the **interval of convergence** of a power series;
3. use the formula for **Taylor** (or **Maclaurin**) **series** to find a power series representation of a function;
4. find the power series representation of a function by manipulating, **differentiating**, **integrating** other power series;
5. use partial sums or the **remainder** of Taylor series to approximate a value of a function or a definite integral to a desired accuracy, and to prove that a function is equal to its power series expansion; and
6. use power series to evaluate limits.

### Potential Learning Approaches and Issues

1. We begin by exploring the general notion of a power series centred at 0,  $\sum_{n=0}^{\infty} c_n x^n$ , where the divergence or convergence of the series depends on the value of  $x$ . The usual tests for convergence apply to power series, but the results will depend on  $x$ . We can move the centre of the series to  $a$  and study similarly  $\sum_{n=0}^{\infty} c_n (x - a)^n$ .
2. Sometimes power series converge for all values of  $x$ .
3. Sometimes they diverge for all values of  $x$  except the centre of the series (0 or  $a$ ).
4. Sometimes power series converge in some open interval that is symmetric around the centre of the series, and we speak of the *radius of convergence* of the series as a way of thinking about how to generate this interval. Finding this radius of convergence, which is determined by the coefficients  $c_n$ , is relatively easy, and we usually use the ratio test as the basis for this task. (Though, in special cases, there are other ways of finding the radius of convergence, none of which we will focus on in this course.) In particular, there is a theorem due to the Norwegian mathematician Abel that says for the series  $\sum_{n=0}^{\infty} c_n (x - a)^n$  the radius of convergence,  $R$ , is given by

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right|.$$

Note that the limit on the right-hand side arises as part of the expression we get when we apply the Ratio Test to this series.

5. As a note of caution about the preceding item, the result should be applied carefully. In the case  $\sum_{n=0}^{\infty} c_n(x-a)^{2n}$ , in which the series only has even powers, the radius of convergence is actually the square root of the value given above. If you think about applying the Ratio Test to this series, then you end up with an  $|x|^2$  in the expression instead of an  $|x|$ , which gives the difference.
6. There is then the question of the convergence of the series at each of the two end points of this interval of convergence, and we test the specific series we get at those end points to determine the convergence there. It need not be the same results at both end points. For example, the series  $\sum_{n=1}^{\infty} (-1)^{n+1} x^n/n$  has a radius of convergence equal to one about 0, and it diverges at  $x = -1$  and converges at  $x = 1$ , giving the *interval of convergence*  $-1 < x \leq 1$  (or  $(-1, 1]$  in interval notation). This example is generated by our old friend  $\ln(1+x)$ , and so the divergence at  $x = 1$  corresponds to attempting to evaluate  $\ln(0)$ .
7. In the interval of convergence of a power series, it is possible to carry out such operations as differentiation and integration meaningfully, and the results of these operations are as you expect, related to the same operations on the functions. Indeed, this is at the heart of the idea that a series can be a *representation* of a function. As you might hope, we are able to carry out *term-by-term* differentiation and integration of series in their interval of convergence and the results are the deviate and antiderivative, respectfully, of the function represented by the series.
8. We will extend the idea from MATH 100 of the Taylor (and Maclaurin) polynomial to Taylor (and Maclaurin) series. This will give us access to series representations for functions such as  $e^x$ ,  $\ln(x)$ ,  $\sin(x)$ ,  $\cos(x)$ , and so on.
9. Recall that the formula for the coefficients for the Taylor series  $\sum_{n=0}^{\infty} c_n(x-a)^n$  is  $c_n = f^{(n)}(a)/n!$ , for  $n = 0, 1, 2, \dots$ . In the case  $a = 0$ , we call the series a Maclaurin series.
10. It is possible to generate the series for products of functions, e.g.  $e^x \sin(x)$ , by multiplying the series for each factor. At the very least, you can easily draw out the first few terms of the series, as needed. In the case where you do algebraic operations with polynomials and series, it can be easy to generate the series: for example, the series for  $e^{x^2}$  is obtained by substituting  $x^2$  into the series for  $e^x$ . The series for  $\ln(1+2x)$  is likewise obtained by substituting  $2x$  into the series for  $\ln(1+x)$ , but what is the radius of convergence of this new series given that the radius of convergence for  $\ln(1+x)$  is 1? Compute it to see what happens.
11. We will be able to use these series to do things like compute “difficult” limits, like

$$\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2},$$

or find approximations to integrals, like  $\int_0^1 e^{-x^2} dx$ . The main point is that the series represent the functions to which they are related, and so it is often just a matter of understanding what happens if we look at the first few terms of the series explicitly in a calculation. Stewart does both of the examples I have given here in section 11.10.

## Suggested Problems

Section 11.8: 3, 5, 7, 17, 20, 38, 41, 42.

Section 11.9:1, 2, 5, 11, 16, 25, 27, 31, 32, 34, 37, 39.

Section 11.10: 1, 2, 7, 8, 13, 19, 31, 33,38, 47, 50, 56, 57, 59, 62, 65, 69.