

MATH 101: Week 7 Learning Goals

February 23, 2014

This week has two main topics: numerical integration (section 7.7) and moments, centres of mass, and centroids (in section 8.3).

Learning Goals

The specific learning goals for this week are that by the end of the week and review homework, you should be able to:

1. Approximate the value of a definite integral using the **midpoint rule**, the **trapezoidal rule**, and **Simpson's rule** to a desired accuracy.
2. Compare the errors in using the Midpoint, Trapezoid, and Simpson's Rules to approximate integrals. You should be able to estimate the improvement in accuracy you would get by doubling the number of subintervals used in each case.
3. Construct an integral or a sum of integrals that can be used to find the **moment** about the x - or y - axis and the x - or y -coordinate of the **centroid** of a plane region.

Potential Learning Approaches and Issues

1. Numerical integration is commonly used in two key situations: (1) you are integrating a function for which there is no elementary closed form antiderivative (e.g. $f(x) = e^{-x^2}$), and (2) you are given a table of values for a function instead of its closed functional form (e.g. computing enthalpies from measured heat capacity data in a physical chemistry laboratory).
2. In the Trapezoid Rule, you are approximating the function by a sequence of connected line segments. Interestingly, this is slightly worse, theoretically, than using the function value at the midpoint of each subinterval (as you do in the Midpoint Rule) by a constant factor of 2. Figure 5 (on page 509) in section 7.7 illustrates why this is so, and it is worth studying this figure until you understand it.
3. Suppose you choose to divide the interval over which you are integrating into an even number n of subintervals. If you look at the formulas for the Trapezoid Rule and Simpson's Rule, you will notice that the same data – the function values at the endpoints of these subintervals – appear in both formulas, but the weights they have in the calculation (these are the coefficients in front of these function values) are different. This illustrates that you can gain improvements in approximating the definite integral for a given function over a given interval by your choice of numerical integration method.
4. For the most part, you will work with functions that do not oscillate rapidly over small intervals, so you can focus on the size of the subintervals, $h = (b - a)/n$, when thinking about how the error improves when you increase the number of subintervals. In fact, you can often just focus on the number of subintervals to compare methods: e.g. doubling the number of subintervals will improve the result for the Trapezoid Rule by a factor of 1/4, whereas if you use Simpson's Rule, you will improve the error by a factor of 1/16. This is related to the fact that n^2 appears in the denominator for the error in the Trapezoid Rule and n^4 appears in the denominator for the error for Simpson's Rule.

5. in section 8.3, we are only interested in the material under Moments and Centres of Mass that starts at the bottom of page 554 and goes to the middle of page 559. We will not do Pappus' Theorem, which starts at the middle of page 559.
6. In your first pass through section 8.3, it may help to focus on the case of a set of point masses arranged along a straight line. This will involve simple sums instead of integrals, and you can work on understanding the basic principles you need to master to work with continuous mass distributions. In the case of point masses distributed along a straight line, the centre of mass is the place where you would put a fulcrum to balance the masses. Start with the case of two, unequal masses, one at each end of massless rod. At what point on the rod would you balance the rod so that it did not rotate?
7. For masses arranged along a straight line, the moment of a mass relative to some fixed point on the line is a measure of the tendency of that mass to cause rotation around that fixed point. The first thing to note is that the point you fix to compute moments is arbitrary, and you will get different values for the same mass by changing this fixed point. Often this fixed point is chosen to be the origin of a coordinate system. (In practice, in a physical problem the fixed point may be a given to you.) In the centre of mass problem, you are looking to find a point relative to which the sum of the tendencies to rotate is zero. (Note that there is a sign to moments.) If you are given a coordinate system, then the problem is to find the x -value of this centre of mass.
8. Instead of looking at moments relative to a fixed point, it is often useful to look at moments relative to a fixed axis. In this case, the centre of mass is then given by a coordinate in two dimensions. These two centre of mass coordinates are found by considering the distributions of the masses along the x direction and along the y direction. That is, *separately* imagine that the point particles are distributed along the x -axis and along the y -axis and compute the centres of mass for each of these situations. The coordinates you get for \bar{x} and \bar{y} give you the centre of mass coordinates in the (x, y) -plane for the distribution of masses.
9. There are two generalizations to these problems that involve integration: (1) consider a continuous distribution of masses along a straight line (e.g. consider a solid rod with some mass distribution) and (2) consider having a distribution of mass over a plane region. The principle is the same as for the point masses, except that you will need to set up a contribution to the calculation for such "mass element" dm and then integrate. Be sure that you understand how to compute the centre of mass for the linear mass distribution before proceeding to do them for planar mass distributions.
10. To some extent, one of the things that makes this section difficult is the interplay between coordinate systems and the more natural concept of centre of mass. In many cases in physical problems, we want to find the centre of mass in order to have a natural origin for a coordinate system in which other computations become easy. Again, it may be best to think of this issue first in the case of a distribution of point masses along a straight line – start with some arbitrary origin and find the centre of mass. Then change coordinates so that this centre of mass becomes the origin and think about how you would have found the centre of mass had you started with this coordinate system.

Suggested Problems

Section 7.7: 1, 2, 7, 9, 12, 17, 22, 29, 30, 32 (repeat this using the Trapezoid Rule instead).

Section 8.3: 21, 22,23, 25, 27, 34, 39*, 41.