24 Feb. 2014

Comparison Theorem

Does $\int_0^\infty \frac{1}{\sqrt{x^3+1}}\,dx$ converge?

$\frac{1}{\sqrt{x^3+1}}$ "looks like" $\frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}}$

But this has a singularity at $x=0$

So $\int_0^\infty \frac{1}{\sqrt{x^3+1}}\,dx = \underbrace{\int_0^1 \frac{1}{\sqrt{x^3+1}}\,dx}_{\text{finite since } \frac{1}{\sqrt{x^3+1}} \text{ continuous}} + \underbrace{\int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx}_{\text{focus on this}}$

Is $\frac{1}{\sqrt{x^3+1}} < \frac{1}{x^{3/2}}$ on $[1,\infty)$?

This is the same as asking:

Is $\sqrt{x^3+1} > x^{3/2}$ on $[1,\infty)$?

$\Leftrightarrow \sqrt{x^3+1} > \sqrt{x^3}$

$\Downarrow$

TRUE since $x^3+1 > x^3$

Thus, we know

$$\frac{1}{\sqrt{x^3+1}} < \frac{1}{x^{3/2}} \text{ on } [1,\infty).$$

$$\Rightarrow \int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx < \underbrace{\int_1^\infty \frac{1}{x^{3/2}}\,dx < \infty}.$$

This converges.

$$\Rightarrow \int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx \quad \underline{\text{CONVERGES}}$$

by Comparison Thm. ∎

Example:

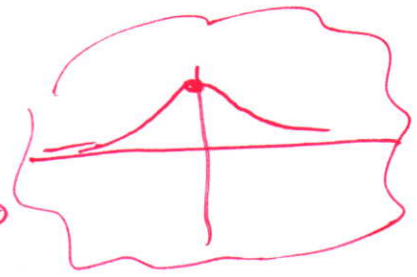
$$\int_1^\infty \frac{\sin(x^2)}{1+x^2}\,dx$$

Consider $\dfrac{\sin(x^2)}{1+x^2}$ on $[1,\infty)$.

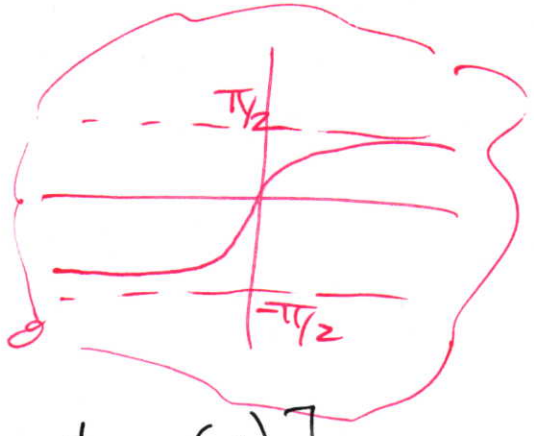
$$-\frac{1}{1+x^2} \le \frac{\sin(x^2)}{1+x^2} \le \frac{1}{1+x^2} \quad \text{on } [1,\infty)$$

$$\Rightarrow \underbrace{-\int_1^\infty \frac{1}{1+x^2}\,dx}_{\text{CONVERGES.}} \le \int_1^\infty \frac{\sin(x^2)}{1+x^2}\,dx \le \underbrace{\int_1^\infty \frac{1}{1+x^2}\,dx}_{\text{CONVERGES}}$$

Let's compute $\int_1^\infty \frac{1}{1+x^2}\,dx$



$$= \lim_{R\to\infty} \int_1^R \frac{1}{1+x^2}\,dx$$



$$= \lim_{R\to\infty} \arctan(x)\Big|_1^R$$

$$= \lim_{R\to\infty} [\arctan(R) - \arctan(1)]$$

$$= \pi/2 - \pi/4 = \pi/4 .$$

Thus,

$$-\pi/4 \le \int_1^\infty \frac{\sin(x^2)}{1+x^2}\,dx \le \pi/4 .$$

FROM the Mock Midterm.

$$f(x) = 4x - \int_0^x f(t)\,dt$$

Find f:

$$f'(x) = 4 - f(x)$$

$$\Rightarrow \boxed{y' = 4 - y} \qquad y = f(x)$$

Let $u = 4 - y \Rightarrow u' = 0 - y' = -y'$

$$\Rightarrow y' = -u'$$

$$-u' = u$$

$$\Rightarrow u' = -u$$

$$u(x) = Ce^{-x}$$

$$\Rightarrow 4 - y = Ce^{-x}$$

$$\Rightarrow 4 - f(x) = Ce^{-x}$$

$$\Rightarrow f(x) = 4 - Ce^{-x}$$

Now $f(x) = 4x - \int_0^x f(t)\,dt$

$\Rightarrow f(0) = 0 - \int_0^0 f(t)\,dt = 0$

$\Rightarrow f(0) = 4 - C = 0$

$\Rightarrow C = 4$

$\Rightarrow \boxed{f(x) = 4 - 4e^{-x}}$