First Name: ________________________  Last Name: ________________________
Student-No: ________________________  Section: ________________________

Grade:

The remainder of this page has been left blank for your workings.
1. **2 marks** Each part is worth 1 mark. Please write your answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

(a) Suppose \( g(x) \) is a function such that \( g(1) = 3 \) and \( g'(x) \geq 5 \) for all \( x \). What is the smallest value that \( g(5) \) could possibly be?

**Answer:** 23

**Solution:** By the mean value theorem, there exists a number \( c \) in \((1, 5)\) such that
\[
g'(c) = \frac{g(5) - g(1)}{5 - 1} = \frac{g(5) - 3}{4}.
\]
We know \( g'(c) \geq 5 \), and so
\[
\frac{g(5) - 3}{4} \geq 5 \implies g(5) - 3 \geq 20 \implies g(5) \geq 23.
\]

(b) Determine the interval(s) on which \( f(x) = \frac{1}{3}x^4 - 2x^3 + 7x + 10 \) is concave up.

**Answer:** \((-\infty, 0)\) and \((3, \infty)\)

**Solution:** We have
\[
f'(x) = \frac{4}{3}x^3 - 6x^2 + 7 \quad \text{and} \quad f''(x) = 4x^2 - 12x.
\]
Now, \( f''(x) = 0 \) when \( x = 0 \) and \( x = 3 \); these are the possible points of inflection for \( f \). On \((-\infty, 0)\), \( f''(x) = 4x(x - 3) > 0 \) (product of two negative numbers), so \( f \) is concave up. On \((0, 3)\), \( 4x(x - 3) < 0 \) \((x - 3 < 0 \text{ while } x > 0)\), so \( f \) is concave down. On \((3, \infty)\), \( 4x(x - 3) > 0 \), so \( f \) is concave up.
Short answer questions — you must show your work

2. [4 marks] Each part is worth 2 marks.
(a) Find the critical points and singular points of \( f(x) = (x^3 + 9x^2)^{1/3} \).

**Solution:** We have
\[
f'(x) = \frac{1}{3} (x^3 + 9x^2)^{-2/3} (3x^2 + 18x) = \frac{3x^2 + 18x}{3(x^3 + 9x^2)^{2/3}}.
\]
The denominator is undefined whenever \( x^3 + 9x^2 = 0 \), i.e. \( x^2(x + 9) = 0 \); these are the singular points for \( f(x) \).
We have \( 3x^2 + 18x = 0 \) when \( x = 0 \) or \( x = -6 \); but since \( f'(0) \) is undefined, \( x = -6 \) is the only critical point for \( f(x) \).

**Marking scheme:** 1 partial mark available for:
- correct derivative AND identifying at least one point
- incorrect derivative, but correctly identifying all critical/singular points based on the incorrect derivative

(b) Determine the intervals on which \( f(x) = (x^3 + 9x^2)^{1/3} \) is increasing or decreasing. You may use your work from part (a).

**Solution:** We check the sign of \( f'(x) = \frac{3x(x + 6)}{3(x^3 + 9x^2)^{2/3}} \) on the intervals determined by the critical and singular points from part (a).
First, note that the denominator is always positive; it’s three times a square.
On \((-\infty, -9)\), both \( 3x \) and \( x + 6 \) are negative, so \( f'(x) > 0 \) and hence \( f(x) \) is increasing.
On \((-9, -6)\), both \( 3x \) and \( x + 6 \) are negative, so \( f'(x) > 0 \) and hence \( f(x) \) is increasing.
On \((-6, 0)\), \( 3x \) is negative while \( x + 6 \) is positive, so \( f'(x) < 0 \) and hence \( f(x) \) is decreasing.
On \((0, \infty)\), we have \( f'(x) > 0 \), and so \( f(x) \) is increasing.

**Marking scheme:** Full marks if work for this part is completely correct, even if work from part (a) was incorrect (exception: incorrect work in part (a) trivializes the problem. In this case, at most one mark). 1 mark if increase/decrease correctly identified for all but one interval; 0 marks otherwise.
Long answer question — you must show your work

3. 4 marks You are designing a box with a **square base**. The volume of the box must be 12 cubic meters. The material that will be used on the sides and top of the box costs $1 per square meter; the bottom must be made of stronger material, which costs $2 per square meter. What should the dimensions of the box be in order to minimize cost?

(Hint: Each of the six sides of this box is a rectangle, so the total surface area of the box is the sum of the areas of these rectangles.)

**Solution:** Call the length of a side of the base of the box $x$ (in meters), and call the height of the box $y$ (in meters). The cost of the base is then $2x^2$, and the cost of the other five sides is $x^2 + 4xy$. Hence, we want to maximize $C = 3x^2 + 4xy$, subject to the constraint $12 = x^2y \implies y = \frac{12}{x^2}$. Thus we want to maximize

$$C(x) = 3x^2 + 4x \frac{12}{x^2} = 3x^2 + \frac{48}{x}$$

on the interval $(0, \infty)$.

We have

$$C'(x) = 6x - \frac{48}{x^2}.$$  

We have $C'(x) = 0$ when $6x - \frac{48}{x^2} = 0 \implies 6x = \frac{48}{x^2} \implies x^3 = 8 \implies x = 2$.

If $0 < x < 2$, then $C'(x)$ has the same sign as $C'(1) = 6 - 48 < 0$; therefore, $C(x)$ is decreasing on $0 < x < 2$. If $x > 2$, then $C'(x)$ has the same sign as $C'(3) = 18 - \frac{48}{9} > 0$; therefore, $C(x)$ is increasing on $2 < x < \infty$. So $C(x)$ has a minimum at $x = 2$; since this is the only critical point, it must be the global minimum. So the dimensions of the box should be $x = 2, y = 3$.

**Marking scheme:**

- 1 mark for $C(x)$
- 1 mark for stating or using the constraint equation $12 = x^2y$
- 1 mark for finding critical numbers of the $C(x)$ the student finds
- 1 mark for justifying that the minimum occurs where the student says it does