First Name: ___________________________ Last Name: ___________________________
Student-No: ___________________________ Section: ___________________________

Grade:

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Very short answer questions

1. [2 marks] Each part is worth 1 marks. Please write your answers in the boxes. Marking scheme: 1 for each correct, 0 otherwise

(a) Suppose $g(x)$ is a function such that $g(1) = 7$ and $g'(x) \leq 2$ for all $x$. What is the largest value that $g(5)$ could possibly be?

Answer: 15

Solution: By the mean value theorem, there exists a number $c$ in $(1, 5)$ such that

$$g'(c) = \frac{g(5) - g(1)}{5 - 1} = \frac{g(5) - 7}{4}.$$ 

We know $g'(c) \leq 2$, and so

$$\frac{g(5) - 7}{4} \leq 2 \implies g(5) - 7 \leq 8 \implies g(5) \leq 15.$$ 

(b) Determine the interval(s) on which $f(x) = \frac{1}{3}x^4 + \frac{4}{3}x^3 + 11x$ is concave up.

Answer: $(-\infty, -2)$ and $(0, \infty)$

Solution: We have

$$f'(x) = \frac{4}{3}x^3 + 4x^2 + 11 \quad \text{and} \quad f''(x) = 4x^2 + 8x.$$ 

Now, $f''(x) = 0$ when $x = 0$ and $x = -2$; these are the possible points of inflection for $f$. On $(-\infty, -2)$, $f''(x) = 4x(x + 2) > 0$ (product of two negative numbers), so $f$ is concave up. On $(-2, 0)$, $4x(x + 2) < 0$ ($x < 0$ while $x + 1 > 0$), so $f$ is concave down. On $(0, \infty)$, $4x(x + 2) > 0$, so $f$ is concave up.
Short answer questions — you must show your work

2. [4 marks] Each part is worth 2 marks.

(a) Find the critical points and singular points of \( f(x) = (x^3 + 6x^2)^{1/3} \).

**Solution:** We have

\[
f'(x) = \frac{1}{3} (x^3 + 6x^2)^{-2/3} (3x^2 + 12x) = \frac{3x^2 + 12x}{3(x^3 + 6x^2)^{2/3}}.
\]

The denominator is undefined whenever \( x^3 + 6x^2 = 0 \), i.e. when \( x^2(x + 6) = 0 \), i.e. at \( x = 0 \) and \( x = -6 \); these are the singular points for \( f(x) \).

We have \( 3x^2 + 12x = 0 \) when \( x = 0 \) or \( x = -4 \); but since \( f'(0) \) is undefined, \( x = -4 \) is the only critical point for \( f(x) \).

**Marking scheme:** 1 partial mark available for:

- correct derivative AND identifying at least one point
- incorrect derivative, but correctly identifying all critical/singular points based on the incorrect derivative

(b) Determine the intervals on which \( f(x) = (x^3 + 6x^2)^{1/3} \) is increasing or decreasing. You may use your work from part (a).

**Solution:** We check the sign of \( f'(x) = \frac{3x(x + 4)}{3(x^3 + 6x^2)^{2/3}} \) on the intervals determined by the critical and singular points from part (a).

First, note that the denominator is always positive; it’s three times a square.

On \( (-\infty, -6) \), both \( 3x \) and \( x + 4 \) are negative, so \( f'(x) > 0 \) and hence \( f(x) \) is increasing.

On \( (-6, -4) \), both \( 3x \) and \( x + 4 \) are negative, so \( f'(x) > 0 \) and hence \( f(x) \) is increasing.

On \( (-4, 0) \), \( 3x \) is negative while \( x + 4 \) is positive, so \( f'(x) < 0 \) and hence \( f(x) \) is decreasing.

On \( (0, \infty) \), we have \( f'(x) > 0 \), and so \( f(x) \) is increasing.

**Marking scheme:** Full marks if work for this part is completely correct, even if work from part (a) was incorrect (exception: incorrect work in part (a) trivializes the problem. In this case, at most one mark). 1 mark if increase/decrease correctly identified for all but one interval; 0 marks otherwise.
Long answer question — you must show your work

3. [4 marks] You are designing a box with a square base. The volume of the box must be 12 cubic meters. The material that will be used on the sides and top of the box costs $1 per square meter; the bottom must be made of stronger material, which costs $2 per square meter. What should the dimensions of the box be in order to minimize cost?

(Hint: Each of the six sides of this box is a rectangle, so the total surface area of the box is the sum of the areas of these rectangles.)

Solution: Call the length of a side of the base of the box $x$ (in meters), and call the height of the box $y$ (in meters). The cost of the base is then $2x^2$, and the cost of the other five sides is $x^2 + 4xy$. Hence, we want to maximize $C = 3x^2 + 4xy$, subject to the constraint $12 = x^2y \implies y = \frac{12}{x^2}$. Thus we want to maximize

$$C(x) = 3x^2 + 4x \frac{12}{x^2} = 3x^2 + \frac{48}{x}$$

on the interval $(0, \infty)$.

We have

$$C'(x) = 6x - \frac{48}{x^2}.$$ 

We have $C'(x) = 0$ when $6x - \frac{48}{x^2} = 0 \implies 6x = \frac{48}{x^2} \implies x^3 = 8 \implies x = 2$.

If $0 < x < 2$, then $C'(x)$ has the same sign as $C'(1) = 6 - 48 < 0$; therefore, $C(x)$ is decreasing on $0 < x < 2$. If $x > 2$, then $C'(x)$ has the same sign as $C'(3) = 18 - \frac{48}{9} > 0$; therefore, $C(x)$ is increasing on $2 < x < \infty$. So $C(x)$ has a minimum at $x = 2$; since this is the only critical point, it must be the global minimum. So the dimensions of the box should be $x = 2, y = 3$.

Marking scheme:

- 1 mark for $C(x)$
- 1 mark for stating or using the constraint equation $12 = x^2y$
- 1 mark for finding critical numbers of the $C(x)$ the student finds
- 1 mark for justifying that the minimum occurs where the student says it does