First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Grade:

The remainder of this page has been left blank for your workings.
Very short answer questions

1. 2 marks Each part is worth 1 marks. Please write your answers in the boxes.  

Marking scheme: 1 for each correct, 0 otherwise

(a) Find the second-degree Maclaurin polynomial for \( f(x) = 2e^{3x} \).

Answer: \( 2 + 6x + 9x^2 \)

Solution: The second-degree Maclaurin polynomial for a function \( f(x) \) is given by

\[
f(0) + f'(0)x + \frac{f''(0)}{2!}x^2.
\]

For \( f(x) = 2e^{3x} \), we have

\[
f(0) = 2, \quad f'(x) = 6e^{3x} \quad \Rightarrow \quad f'(0) = 6, \quad f''(x) = 18e^{3x} \quad \Rightarrow \quad \frac{f''(0)}{2!} = 9.
\]

So the second-degree Maclaurin polynomial for \( f(x) = 2e^{3x} \) is \( 2 + 6x + 9x^2 \).

(b) Let \( T_4(x) = 14 + 20(x-2) - 6(x-2)^2 + 10(x-2)^3 + 40(x-2)^4 \) be the fourth-degree Taylor polynomial for a function \( h(x) \) about \( x = 2 \). What is \( h^{(3)}(2) \) (that is, the third derivative of \( h(x) \) at \( x = 2 \))?

Answer: \( h^{(3)}(2) = 60 \)

Solution: By the formula for the \( n \)th Taylor polynomial, the coefficient of the \( (x - 2)^3 \) term is given by \( \frac{h^{(3)}(2)}{3!} \). The coefficient of \( (x - 2)^3 \) for the given polynomial is 10, so it must be the case that

\[
\frac{h^{(3)}(2)}{3!} = 10 \quad \Rightarrow \quad h^{(3)}(2) = 10 \cdot 3! = 60.
\]
Short answer questions — you must show your work

2. [4 marks] Each part is worth 2 marks.

(a) Estimate $\sqrt{10}$ using a linear approximation.

**Solution:** Let’s use the first Taylor polynomial for $f(x) = \sqrt{x}$ about $x = 9$, since 9 is close to 10 and we know that $f(9) = 3$. We have

$$T_1(x) = f(9) + f'(9)(x - 9).$$

Now, $f'(x) = \frac{1}{2\sqrt{x}} \implies f'(9) = \frac{1}{6}$. Therefore,

$$f(10) \approx T_1(10) = 3 + \frac{1}{6}(10 - 9) = 3 + \frac{1}{6}.$$

**Marking scheme:** 2 marks for correct answer with good work. 1 partial mark available for: correct general formula for $T_1(x)$ about any point; slightly incorrect formula but correct derivative and subsequent work.

(b) The first Maclaurin polynomial for $f(x) = 2\sin(x)$ is used to estimate $2\sin(0.1)$. Give and justify an upper bound for the absolute error in this approximation.

**Solution:** The Lagrange remainder formula with $x = 0.1$ and $a = 0$ says that

$$|R_1| \leq M \frac{|0.1|^2}{2!}.$$

where $M$ is an upper bound for $|f^{(2)}(c)|$ over $c$ in the interval $(0, 0.1)$. Note that

$$|f^{(2)}(c)| = |-2\sin(c)| = 2|\sin(c)| \leq 2,$$

since $\sin(x) \leq 1$ for all real numbers $x$. Therefore, the absolute error is bounded as follows:

$$|R_1| \leq 2\frac{(0.1)^2}{2} = \frac{2}{200}.$$

**Marking scheme:** 1 mark for an attempt at using the formula; 1 mark for good justification. It’s possible to come up with a different error bound (with justification) and receive full marks.
Long answer question — you must show your work

3. 4 marks A blimp flying in a horizontal line with constant velocity at an altitude of 3 km passes directly above an observer on the ground at 1PM. One hour later, at 2PM, the blimp has traveled 4 km from the point directly above the observer. In kilometers per hour, what is the rate of change of the distance between the blimp and the observer at 2PM?

Solution:

In the diagram to the right, \( D \) is the distance between the blimp and the observer, and \( x \) is the distance traveled by the blimp after it flies over the observer. Based on this notation, we want to know \( \frac{dD}{dt} \) when \( x = 4 \). The Pythagorean theorem tells us that

\[ 3^2 + x^2 = D^2. \]

Differentiating, we obtain

\[ 2x \frac{dx}{dt} = 2D \frac{dD}{dt}. \]

Since the blimp travels 4 km in 1 hour and has constant velocity, we know that \( \frac{dx}{dt} = 4 \). Also, when \( x = 4 \), we have

\[ 3^2 + 4^2 = D^2 \implies 25 = D^2 \implies D = 5. \]

Therefore, when \( x = 4 \), we have

\[ \frac{dD}{dt} = \frac{2 \cdot 8 \cdot 4}{2 \cdot 5} = \frac{64}{10}. \]

So the distance between the observer and the blimp is increasing at a rate of 6.4 km/hr.

Marking scheme:

- 1 mark for starting from correct equation
- 1 mark for correct differentiation with respect to \( t \) (whether equation is correct or not, as long as differentiation is nontrivial)
- 1 mark for a reasonable attempt at identifying unknowns when \( x = 4 \) (i.e. that \( \frac{dx}{dt} = 4 \) and that \( D = 5 \) when \( x = 4 \))
- 1 mark for solving for \( \frac{dD}{dt} \)