Question 1.
A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He does not need a fence along the river. What are the dimensions of the field that has the largest area?

\[ 2x + y = 2400 \]

Area of the field:
\[ A = xy \]
\[ \Rightarrow y = 2400 - 2x \]
\[ \Rightarrow A(x) = x(2400 - 2x), \quad 0 \leq x \leq 1200 \]

Maximize \( A(x) \), i.e., find global maximum on the interval \([0, 1200]\).

\[ A(x) = 2400x - 2x^2 \]
\[ A'(x) = 2400 - 4x \]
\[ 2400 - 4x = 0 \Rightarrow 4x = 2400 \Rightarrow x = 600. \]

No singular pts. Test \( A(x) \) at \( x = 600, 0, 1200 \)
\[ A(0) = 0, \quad A(1200) = 0, \quad A(600) = 600 \times 1200 > 0. \]

When \( x = 600, y = 1200 \). So dimensions of largest field are \( 600' \times 1200' \).
Question 2.
Find two numbers whose difference is 100 and whose product is a minimum.

Let $x$ and $y$ be our two numbers.

\text{WANT:} \quad \text{minimize} \quad P = xy

subject to the constraint \quad x - y = 100.

\quad \Rightarrow \quad x = 100 + y

So \quad P = P(y) = y(100 + y), \quad y \text{ any real #.}

\quad P'(y) = 100 + 2y \Rightarrow 100 + 2y = 0 \quad y = -50.

\quad \Rightarrow \quad x = 50.

Since the graph of $P(y)$ is a parabola, the global minimum must occur at the vertex, i.e. $y = -50$.

So $x = 50$ and $y = -50$ are the desired numbers.
Question 3.
Find the point of the line $6x + y = 9$ that is closest to the point $(-3, 0)$.

To choose a point $(x_0, y_0)$ inside

Want to minimize $D$, subject to the constraint $y_0 = -6x_0 + 9$

$$D = \sqrt{y_0^2 + (x_0 + 3)^2}$$

$$D'(x_0) = \frac{1}{2} \left( (-6x_0 + 9)^2 + (x_0 + 3)^2 \right)^{-\frac{1}{2}} \cdot (2(-6x_0 + 9) + 2(x_0 + 3))$$

$$D'(x_0) = \frac{-12(-6x_0 + 9) + 2(x_0 + 3)}{2 \sqrt{(-6x_0 + 9)^2 + (x_0 + 3)^2}}$$

No singular pts.

Critical pts: $-12(-6x_0 + 9) + 2x_0 + 6 = 0$

$72x_0 - 108 + 2x_0 + 6 = 0$

$74x_0 = 102$

$x_0 = \frac{102}{74}$

So the point on the line closest to $(-3, 0)$ is $(\frac{102}{74}, -6(\frac{102}{74}) + 9)$. 
Question 4.
A cylindrical can is being made to contain 1 L of oil. Find the dimensions that will minimize the amount of metal needed to make the can.

Want to minimize surface area:
\[ SA = 2 \pi r^2 + 2 \pi rh \]

subject to the constraint
\[ \pi r^2 h = 1. \]
\[ h = \frac{1}{\pi r^2} \]

\[ \Rightarrow SA(r) = 2 \pi r^2 + 2 \pi r \left( \frac{1}{\pi r^2} \right), \quad 0 < r \]

Want to minimize this.
\[ SA(r) = 4 \pi r + \frac{2}{r^2} \]

No singular points in the domain \( r > 0 \),
\[ 4 \pi r + \frac{2}{r^2} = 0 \Rightarrow 4 \pi r^3 - 2 = 0 \]
\[ r = \sqrt[3]{\frac{2}{4 \pi}} \]
\[ r = \frac{\sqrt[3]{2}}{\sqrt[3]{4 \pi}} \]

\[ \frac{SA'}{< 0}{< 0} \quad \frac{SA'}{> 0}{> 0} \]

\[ \text{SA decreases at } \frac{1}{\sqrt[3]{4 \pi}} \quad \text{SA increases} \]

\[ \Rightarrow r = \frac{\sqrt[3]{2}}{\sqrt[3]{4 \pi}} \text{ gives the global minimum.} \]

\[ 4 \pi \left( \frac{3 \sqrt[3]{2}}{4 \pi} \right) - 2 \]

\[ < 0 \]

So the dimensions that minimize the amount of metal used are:
\[ r = \frac{3 \sqrt[3]{2}}{4 \pi}, \quad h = \frac{1}{\pi \left( \frac{3 \sqrt[3]{2}}{4 \pi} \right)^2} \]
Question 5.  
If 1200 cm² of material is available to make a box with a square base and open top, find the largest possible volume of the box.

\[
\text{surface area } = A = x^2 + 4xy
\]

We want to maximize \( V = x^2y \)

subject to the constraint \( x^2 + 4xy = 1200 \)

\( \Rightarrow \) solve for \( y \): \( 4xy = 1200 - x^2 \) \( \Rightarrow \) \( y = \frac{1200 - x^2}{4x} \)

So we maximize \( V(x) = x^2 \left( \frac{1200 - x^2}{4x} \right) = \frac{1}{4} x (1200 - x^2) \)

on the interval \( (0, \sqrt{1200}] \)

\( V'(x) = \frac{1}{4} \left( x(-2) + (1200 - x^2) \right) \)

\( = \frac{1}{4} (-3x^2 + 1200) \) \( \Rightarrow \) \( \frac{1}{4} (-3x^2 + 1200) = 0 \)

\(-3x^2 + 1200 = 0 \)

\(-3x^2 = -1200 \)

\( x^2 = 400 \) \( \Rightarrow \) \( x = 20 \)

\( V'(x) > 0 \) \( \Rightarrow \) \( V \) inc. \( 20 \) \( V \) dec

\( V(20) = \frac{1}{4} (-3 + 1200) > 0 \)

\( V'(\sqrt{1200}) = \frac{1}{4} (-3(1200) + 1200) \)

\( = \frac{1}{4} (3) < 0 \)

Check endpoint: \( V(\sqrt{1200}) = 0 \)

So the largest possible volume of the box is \( V(20) = \frac{1}{4} (20)(1200 - 400) = 4000 \text{ cm}^3 \).
**Question 6.**

You stand on a cliff at point \((0,0)\) overlooking a river. You see a boat due north at point \((0,2)\). The boat is traveling down the river along the curve \(y = \sqrt{x + 4}\) towards the harbour at \((-4,0)\). You want to wave to the boat at the point where it is closest to you. Find the coordinates of this point.

\[
\text{Distance from } (x,y) \text{ to } (0,0): \\
D = \sqrt{x^2 + y^2}
\]

want to minimize

\[
D = \sqrt{x^2 + y^2} \quad \text{subject to} \\
\text{constraint} \quad y = \sqrt{x+4}.
\]

\[\Rightarrow D(x) = \sqrt{x^2 + (\sqrt{x+4})^2} = \sqrt{x^2 + x + 4}, \quad -4 \leq x \leq 0\]

\[
D'(x) = \frac{1}{2} \left( x^2 + x + 4 \right)^{-1/2} \cdot \left( 2x + 1 \right) = \frac{2x + 1}{2 \sqrt{x^2 + x + 4}} \\
\text{possible } x \text{-coordinates of } pts \text{ in the graph above}
\]

**singular pts occur when** \(x^2 + x + 4 = 0\) \quad \text{(square root of a negative number)}

\[
x = \frac{-1 \pm \sqrt{1 - 4(1)(4)}}{2} \Rightarrow \text{no solutions to}
\]

**critical pts:** \(2x + 1 = 0\)

\[
x = -\frac{1}{2}
\]

**global min of** \(D(x)\) **occurs at** \(x = 0, x = 1\) or \(x = -\frac{1}{2}\), since \(D(x)\) is continuous on the closed interval \([-4, 0]\).

\[
D(0) = \sqrt{0 + 0 + 4} = 2
\]

\[
D(-4) = \sqrt{16 - 4 + 4} = 4
\]

\[
D\left(-\frac{1}{2}\right) = \sqrt{\frac{1}{4} - \frac{1}{2} + 4} = \sqrt{4 - \frac{1}{4}} < 2 \quad \text{(since } \sqrt{4} = 2)\]

So the point at which the boat is closest to me is

\[
\left( -\frac{1}{2}, \sqrt{\frac{1}{2} + 4} \right). \quad \text{(Recall: } y = \sqrt{x+4})
\]
Question 7.
(Final Exam 2014) Particle 1 moves on the y-axis starting at the point (0, 6) and travels towards the origin with a constant speed of 2 units per second. Particle 2 moves on the x-axis starting at the origin and travels in the positive x-direction with constant speed of 1 unit per second. At what moment in the first 3 seconds is the distance minimized?

\[ D(t) = \sqrt{(6-2t)^2 + t^2}, \quad 0 \leq t \leq 3 \]

Find singular/crit pts, evaluate D(t) at these pts + endpoints; smallest value is the global minimum.

\[ D'(t) = \frac{1}{2} \left( (6-2t)^2 + t^2 \right)^{-\frac{1}{2}} \left( 2(6-2t)(-2) + 2t \right) \] snapshot at time t, \( 0 \leq t \leq 3 \)

\[ D'(t) = \frac{-12(6-2t) + 2t}{2((6-2t)^2 + t^2)^{\frac{1}{2}}} \]

Since \((6-2t)^2\) and \(t^2\) are not zero at the same time, no singular pts.

Critical pts: \(-12(6-2t) + 2t = 0\)
\[-24 + 8t + 2t = 0\]
\[10t = 24, \quad t = \frac{24}{10}\]

\[ D(t) \] is continuous on \([0, 3]\). So, global minimum occurs at either \(t = 0, \ t = 3\), or \(t = \frac{24}{10}\).

\[ D(0) = \sqrt{6^2 + 0^2} = 6 \]
\[ D(3) = \sqrt{0^2 + 3^2} = 3 \]
\[ D\left(\frac{24}{10}\right) = \sqrt{(6 - \frac{24}{5})^2 + \left(\frac{24}{10}\right)^2} = \sqrt{\left(\frac{-6}{5}\right)^2 + \frac{24^2}{100}} = \sqrt{\frac{36}{25} + \frac{24^2}{100}} = \sqrt{\frac{36 + 24^2}{100}} = \sqrt{\frac{36 + 576}{100}} = \sqrt{\frac{612}{100}} = \sqrt{6.12} < 3 \]
Question 8.
(Final Exam 2013) Find the area of the largest rectangle which has two vertices on the x-axis and two vertices lying on the graph of the function $y = 8 - x^2$ with $-\sqrt{8} \leq x \leq \sqrt{8}$. Please justify your answer.

Area of green rectangle
= (base)(height) = (2x)(8-x^2)

Want to maximize
$A(x) = 2x(8-x^2), 0 \leq x \leq \sqrt{8}$

$A'(x) = (2x)(-2x) + 2(8-x^2)
= -4x^2 + 16 - 2x^2
= -6x^2 + 16$

No singular pts. Critical points:
$-6x^2 + 16 = 0 \Rightarrow x = \sqrt{\frac{16}{6}}$

Since $A(x)$ is continuous on $[0, \sqrt{8}]$, the global maximum occurs at $x = 0, x = \sqrt{8}, x = \sqrt{\frac{16}{6}}$.

$A(0) = 0$
$A(\sqrt{8}) = 2\sqrt{8}(8-8) = 0$
$A(\sqrt{\frac{16}{6}}) = 2\sqrt{\frac{16}{6}}(8 - \frac{16}{6}) > 0$

So the largest area is
$A(\sqrt{\frac{16}{6}}) = 2\sqrt{\frac{16}{6}}(8 - \frac{16}{6})$. 
Question 9.
(Final Exam 2010) Cobblestone Engineering has been contracted to build a bridge and a pathway from Calculus City located at point A on the shoreline of a 6 km wide river that runs west to east, to a power plant at point B on the opposite shoreline, 8 km to the east. It costs 40 dollars per km to build a bridge over the river and 20 dollars per km to build a pathway along a shoreline. How should the company proceed in order to minimize the total cost? Be sure to justify that the total cost is indeed a minimum.

Minimize \( C = 40B + 20P \)

Subject to constraint \( B^2 = 6^2 + (8-P)^2 \).

\[
C(p) = 40\sqrt{6^2 + (8-p)^2} + 20P, \quad 0 \leq P \leq 8
\]

\[
C'(p) = 40 \cdot \frac{1}{2} \left( 6^2 + (8-p)^2 \right)^{-\frac{1}{2}} (2(8-p)(-1)) + 20
\]

\[
= \frac{40(-2(8-p))}{2\sqrt{36 + (8-p)^2}} + 20
\]

No singular pts; denominator always positive.

Critical pts:
\[
\frac{-80(8-p)}{2\sqrt{36 + (8-p)^2}} + 20 = 0
\]

\[
-80(8-p) + 40\sqrt{36 + (8-p)^2} = 0
\]

\[
40\sqrt{36 + (8-p)^2} = 80(8-p)
\]
1600(36+(x-p)^2) = 6400(x-p)^2 \quad (0 \text{ both sides})

\[ 36 + (x-p)^2 = 4(x-p)^2 \]

\[ 36 = 3(x-p)^2 \]
\[ 12 = (x-p)^2 \]
\[ x-p = \pm \sqrt{12} \]
\[ p = 8 \pm \sqrt{12} \]

The only critical point in our domain is \( p = 8 - \sqrt{12} \).

Now: Which is smallest? \( \square \) \( C(0), \ C(8), \ C(8-\sqrt{12}) \)?

\[ C(p) = 40 \sqrt{36+(x-p)^2} + 20x \]
\[ C(0) = 40 \sqrt{36+64} = 400 \]
\[ C(8) = 40 \sqrt{36+0} + 160 = 240 + 160 = 400 \]
\[ C(8-\sqrt{12}) = 40 \sqrt{36+(12)} + 20(8-\sqrt{12}) \]
\[ = 40 \sqrt{48} + 160 - 20 \sqrt{12} < 280 + 160 - 60 \]
\[ = 380 < 400 \]

So, the cost is minimized when the length of the path is \( 8 - \sqrt{12} \) km, and the bridge is \( \sqrt{36+12} \) km long.