4.1: Introduction to Antiderivatives

**Question:** Given a function \( f(x) \), find a function \( F(x) \) such that \( F'(x) = f(x) \).

**Def:** A function \( F(x) \) that satisfies

\[
\frac{d}{dx} [F(x)] = f(x),
\]

is called an antiderivative for \( f(x) \).

Notice the "an": If \( F(x) \) is an antiderivative for \( f(x) \), then so is \( F(x) + C \), since \( \frac{d}{dx} [C] = 0 \), for any constant \( C \).

So typically, if we want to find "the most general" antiderivative for \( f(x) \), we write \( F(x) + C \) to represent all possible antiderivatives (the \( C \) represents any constant).
Ex: $f(x) = x^2$. Find the most general antiderivative for $f(x)$.

Sol: Need a function $F(x)$ such that $F'(x) = f(x) = x^2$.

Try $F(x) = x^3$, then $F'(x) = 3x^2$. Need to adjust, to account for the 3; so introduce a constant factor of $\frac{1}{3}$ to $F(x)$.

$F(x) = \frac{1}{3} x^3$. Then $F'(x) = x^2$, great!

So the most general antiderivative for $f(x) = x^2$ is $F(x) = \frac{1}{3} x^3 + C$, where $C$ is my constant.

Ex: Given: $g'(t) = 6t^2$ and $g(2) = 1$. Find $g(t)$.

Sol: Let's find a function $g(t)$ s.t. $g'(t) = 6t^2$.

$g(t) = \frac{6}{3} t^3 + C$. What is $C$, given $g(2) = 1$. 


Ex: Given \( g'(t) = 6t^4 \) and \( g(2) = 1 \), find \( g(t) \).

Sol: Let's find \( g(t) \) so that \( g'(t) = 6t^4 \).

\[ g(t) = \frac{6}{5} t^5 + C. \]

What is \( C \), given \( g(2) = 1 \)?

\[ 1 = g(2) = \frac{6}{5} 2^5 + C \]

\[ \Rightarrow C = 1 - \frac{6}{5} 2^5. \]

So \( g(t) = \frac{6}{5} t^5 + 1 - \frac{6}{5} 2^5 \).

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**Table of Antiderivatives**

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<th>( x^n ) (( n \neq -1 ))</th>
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<th>( f(x) = F'(x) )</th>
<th>( \sec^2(x) )</th>
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<td>( e^x + C )</td>
<td>( \log(x) + C )</td>
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etc.
Ex: Find the most general antiderivative for
\[ f(x) = \sin(x) + \cos(2x). \]

Sol: Find an antiderivative for \( \sin(x) \) and \( \cos(2x) \) separately, and add them together.

\[ \Rightarrow \quad F(x) = -\cos(x) + \frac{1}{2} \sin(2x) + C. \]

Check: \( F'(x) = \sin(x) + \cos(2x) \)

Ex: Find the most general antiderivative for
\[ g(x) = \frac{1}{1+4x^2} = \frac{1}{1+(2x)^2}. \]

Sol: Notice that
\[ \frac{d}{dx} (\arctan(2x)) = \frac{1}{1+(2x)^2} \cdot 2. \]

So:
\[ G(x) = \frac{1}{2} \arctan(2x) + C. \]
Ex: Find an antiderivative for \( f(x) = \frac{1}{1 + x^2 + 2x} \).

So: \( f(x) = \frac{1}{x^2 + 2x + 1} = \frac{1}{(x+1)^2} = (x+1)^{-2} \).

Try: \( F(x) = (x+1)^{-1} \), \( F'(x) = -(x+1)^{-2} \), no good.

Try again: \( F(x) = -(x+1)^{-1} \), \( F'(x) = \) \( (x+1)^{-2} \) \( \checkmark \)

So \( F(x) = \frac{-1}{x+1} \) is an antiderivative for \( f(x) \).

Ex: Suppose \( Q(t) \) is the amount of a isotope in a sample. Suppose the sample loses \( 50e^{-5t} \) mg per second to decay. If \( Q(1) = 10e^{-5} \) find an equation for the amount of isotope at time \( t \).

So: Given: \( \frac{dQ}{dt} = -50e^{-5t} \) and \( Q(1) = 10e^{-5} \), want to find \( Q(t) \).
Try: \( Q(t) = e^{-st} \) \( \Rightarrow \) \( Q'(t) = -5e^{-st} \), not quite.

Introduce a factor of 10:
\[
Q(t) = 10e^{-st}, \quad \Rightarrow \quad Q'(t) = -50e^{-st}.
\]

\( Q(t) = 10e^{-st} + C \) has the derivative property we want. Also: \( Q(1) = 10e^{-s} \), so:
\[
10e^{-s} = Q(1) = 10e^{-s} + C
\]

\( \Rightarrow \) \( C = 0 \).

So, \( Q(t) = 10e^{-st} \).
Ex: Suppose $f'(t) = 2t + 7$.

What is $f(10) - f(3)$?

Sol: An antiderivative for $f'(t) = 2t + 7$ is

$f(t) = t^2 + 7t + C$, $C$ a constant.

So:

$f(10) - f(3) = 10^2 + 70 + C - (9 + 0 + 21 + C)$

$= 100 + 70 - 9 - 21 = 140$.

Ex: (Final 2012) Find a function $f(x)$ that satisfies $f'(x) = 2 \cos(x) - e^x$ and $f(0) = 0$.

Sol:

$f(x) = 2 \sin(x) - e^x + C$

Check: $f'(x) = 2 \cos(x) - e^x \checkmark$

Now

$0 = f(0) = 2 \sin(0) - e^0 + C$

$\Rightarrow 0 = -1 + C \Rightarrow C = 1$.

$\Rightarrow f(x) = 2 \sin(x) - e^x + 1$. 
Ex: Suppose the acceleration of an object at time $t$ is given by $a(t) = 7$. The initial velocity of the object is 100 km/h, and the position of the object is 4 km/h. Find the position function for this object.

Sol: Know: $s''(t) = v'(t) = a(t)$.

Given $a(t) = 7$, $v(t) = 7t + C$.

Since $v(0) = 100$, we see that $C = 100$.

$\Rightarrow v(t) = 7t + 100$.

So: $s(t) = \frac{7}{2} t^2 + 100t + D$

$\Rightarrow s(0) = 4 \Rightarrow \frac{7}{2} \cdot 0 + 100 \cdot 0 + D = 4$

$\Rightarrow D = 4$.

So $s(t) = \frac{7}{2} t^2 + 100t + 4$. 