Two more MVT examples

1. Let \( f(x) \) be a function such that \( f(1) = 10 \) and \(-1 \leq f'(x) \leq 2\) for all \( x \). Obtain upper and lower bounds for \( f(5) \).

Solution: By the MVT, there exists \( c \) between 1 and 5 such that \( f'(c) = \frac{f(5) - f(1)}{5 - 1} \)

\[
f'(c) = \frac{f(5) - 10}{4}
\]

Since \(-1 \leq f'(c) \leq 2\) \(\Rightarrow\) \(-1 \leq \frac{f(5) - 10}{4} \leq 2\)

\(-4 \leq f(5) - 10 \leq 8\) \(\Rightarrow\) \(6 \leq f(5) \leq 18\)
2) Prove that \( f(x) = x^3 + x + 1 \) has at most one real root.

To the contrary,

\[ S_0 \]: Assume \( f(x) \) has two roots, i.e., there exist two numbers \( a, b \) s.t. \( f(a) = f(b) = 0 \).

\( \Rightarrow \) there exists a point \( c \) between \( a \) and \( b \) s.t. \( f'(c) = 0 \). (MVT) (or Rolle's)

But \( f'(x) = 3x^2 + 1 > 0 \).

\( \Rightarrow f(x) \) has no critical \#s.

But local minima and/or maxima must occur at critical \#s.

So this point \( c \) not not exist \( \Rightarrow \) there can't be two roots.
3.6 Sketching Graphs

G-DAL: Use properties of $f(x)$, $f'(x)$, and $f''(x)$ to sketch an accurate graph of $y=f(x)$.

1) $f(x)$ itself tells you:

- **Domain**: Where is $f(x)$ defined?
- **Intercepts**: Where does $y=f(x)$ cross the axes?
  - *x-intercepts*: Solve $f(x)=0$
  - *y-intercepts*: $f(0)$
- **Vertical asymptotes**: e.g., if $f(x)$ is a rational function, then $y=f(x)$ has vertical asymptotes at the zeros of the denominator.
- **Horizontal asymptotes/end behavior**: the limits
  \[ \lim_{x \to \infty} f(x) \text{ and } \lim_{x \to -\infty} f(x) \]
tell you how $f(x)$ behaves for large values of $x$.  

Ex: \( f(x) = \frac{x+1}{(x+3)(x-2)} \)

**Domain:** all reals except \( x = 2, x = -3 \).

**Intercepts:**
- **X-intercepts:** \( \frac{x+1}{(x+3)(x-2)} = 0 \) \( \Rightarrow \) \( x = -1 \)
- **Y-intercept:** \( f(0) = \frac{1}{(3)(-2)} = -\frac{1}{6} \) ∈ y-intercept.

**Vertical Asymptotes:** For \( x \) near 2:
- if \( x > 2 \): \( \frac{x+1}{(x+3)(x-2)} = \frac{(+)}{(+)} \). So \( \lim_{x \to 2^+} f(x) = \infty \)
- if \( x < 2 \): \( \frac{x+1}{(x+3)(x-2)} = \frac{(+)}{(-)} \). So \( \lim_{x \to 2^-} f(x) = -\infty \)

For \( x \) near -3
- if \( x < -3 \): \( \frac{x+1}{(x+3)(x-2)} = \frac{(-)}{(-)} \). So \( \lim_{x \to -3^-} f(x) = -\infty \)
- if \( x > -3 \): \( \frac{x+1}{(x+3)(x-2)} = \frac{(-)}{(+)} \). \( \lim_{x \to -3^+} f(x) = \infty \)

**End behavior:**
- \( \lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 0 \)
2. \( f'(x) \) tells you:

- **increasing/decreasing**: if \( f'(x) > 0 \) for all \( A \leq x \leq B \), then \( f(x) \) is increasing on \([A, B]\).

- if \( f'(x) < 0 \) for \( A \leq x \leq B \), then \( f(x) \) is decreasing on \([A, B]\).

- **extrema**: can occur at critical pts (\( f'(x) = 0 \)) or singular points (\( f'(x) \) DNE). Global extrema can occur at endpoints, if applicable.

**Ex**: \( f(x) = x^4 - 6x^3 \).

- **Domain**: all reals. No asymptotes.

- **Integrals**: \( x^4 - 6x^3 = 0 \) \( \Rightarrow \) \( x^3(x - 6) = 0 \) \( \Rightarrow \) \( x = 0, x = 6 \).

- **y-intercepts**: \( f(0) = 0 \).

- **End behavior**: \( f(x) = x^4 \left(1 - \frac{6x^3}{x^4}\right) \), so \( f(x) \to \infty \) as \( x \to \pm \infty \).

Now look at \( f'(x) = 4x^3 - 18x^2 \):

- \( f'(x) = 0 \) \( \iff \) \( 4x^3 - 18x^2 = 0 \) \( \iff \) \( x^2(4x - 18) = 0 \)

  - \( x = 0, x = \frac{9}{2} \) \( \in \) critical pts.
<table>
<thead>
<tr>
<th>Interval/Point</th>
<th>$(-\infty, 0)$</th>
<th>$0$</th>
<th>$(0, \frac{a}{2})$</th>
<th>$\frac{a}{2}$</th>
<th>$(\frac{a}{2}, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(x)$</td>
<td>negative</td>
<td>0</td>
<td>negative</td>
<td>0</td>
<td>positive</td>
</tr>
<tr>
<td>$f(x)$</td>
<td>decreasing</td>
<td>horiz.</td>
<td>decreasing</td>
<td>local min.</td>
<td>increasing</td>
</tr>
</tbody>
</table>

$f'(1) = 1(4-18) < 0$
$f'(5) = 25(20-18) > 0$
\[ f''(x) \text{ tells you: Concavity} \]

- Concave Up
  - \( y = f(x) \) lies above all of its tangent lines.

- Concave Down
  - \( y = f(x) \) lies below all of its tangent lines.

\[ g(x) \]
Suppose $f(x)$ is continuous on $[a, b]$, and that $f'(x)$ and $f''(x)$ exist on this interval.

- If $f''(x) > 0$ for all $x$ in $(a, b)$, then the graph of $f$ lies above its tangent lines and is said to be \underline{concave up}.

- If $f''(x) < 0$ for all $x$ in $(a, b)$, then the graph of $f$ lies below its tangent lines, and it is said to be \underline{concave down}.

- If $f''(c) = 0$ for some $c$ between $a$ and $b$, and if the concavity of $f$ changes across $x = c$, then $(c, f(c))$ is called a \underline{inflection point}. 

\[\text{Diagram of concave up and concave down with inflection point.}\]
Ex: Determine the concavity of \( f(x) = x^4 - 6x^3 \).

So: \( f'(x) = 4x^3 - 18x^2 \)

\( f''(x) = 12x^2 - 36x \)

Possible points of inflection: \( 12x^2 - 36x = 0 \)
\( x(12x - 36) = 0 \)
\( x = 0 \) or \( x = 3 \).

\[
\begin{array}{c|c|c|c|c|c}
\text{Interval/pt.} & (-\infty, 0) & 0 & (0, 3) & 3 & (3, \infty) \\
\hline
f''(x) & \text{positive} & 0 & \text{negative} & 0 & \text{positive} \\
\hline
f(x) & \text{concave up} & \text{inf. pt.} & \text{concave down} & \text{inf. pt.} & \text{concave up} \\
\end{array}
\]

\( f''(-1) = (-1)(-12 - 36) > 0 \)

\( f''(1) = (1)(12 - 36) < 0 \)

\( f''(4) = 4(48 - 36) > 0 \)

\( f(3) = 3^4 - 6(3) \)
\( = 81 - 18 = 63 \)

\( f(\frac{3}{2}) \)

\( f(\frac{3}{2}) \)

\( f(\frac{3}{2}) \)
Symmetry

**Def:** A function $f$ is called **even** if $f(-x) = f(x)$ for all $x$.

**Def:** A function $f$ is called **odd** if $f(-x) = -f(x)$.

**Examples:**

<table>
<thead>
<tr>
<th>Even Functions</th>
<th>Odd Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2$</td>
<td>$x^3$</td>
</tr>
<tr>
<td>$1$</td>
<td>$x$</td>
</tr>
<tr>
<td>$1$</td>
<td>$e^{-x}$</td>
</tr>
<tr>
<td>$\cos(x)$</td>
<td>$\sin(x)$</td>
</tr>
</tbody>
</table>

**Notice:** (1) If $f(x)$ is even and $(x_0, y_0)$ is a point on the graph of $y = f(x)$, then:

$f(-x_0) = f(x_0) = y_0$

$\Rightarrow (-x_0, y_0)$ is also a point on $y = f(x)$

$\Rightarrow$ graph is symmetric across the y-axis.
2) If $f(x)$ is odd and $(x_0, y_0)$ is on the graph of $y = f(x)$,

$$f(-x_0) = -f(x_0) = -y_0.$$  

$\Rightarrow$ $(x_0, -y_0)$ is on the graph of $y = f(x)$  

$\Rightarrow$ graph is "symmetric about the origin"
Ex: \[ g(x) = \frac{x^2 - 9}{x^2 + 3} \] (Note: \( g(x) \) is even)

Domain: all reals

Intercepts: \[ \frac{x^2 - 9}{x^2 + 3} = 0 \iff x^2 - 9 = 0 \iff x = \pm 3, \]

\[ g(0) = \frac{-9}{3} = -3 \]

No vert. asympt.

Horizontal asymptotes: \[ \lim_{x \to \infty} g(x) = \lim_{x \to -\infty} g(x) = 1 \]

\[ g'(x) = \frac{2x(x^2 + 3) - 2x(x^2 - 9)}{(x^2 + 3)^2} \quad \text{No sing. pts.} \]

Critical pts: \[ 2x(x^2 + 3) - 2x(x^2 - 9) = 0 \]

\[ 2x(3x^2 + 3 - (x^2 - 9)) = 0 \]

\[ 2x(12) = 0 \iff x = 0 \]

<table>
<thead>
<tr>
<th>interval/pt</th>
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<th>0</th>
<th>((0, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(g(x))</td>
<td>neg.</td>
<td>0</td>
<td>pos.</td>
</tr>
<tr>
<td>(g'(x))</td>
<td>dec.</td>
<td>local min.</td>
<td>inc.</td>
</tr>
</tbody>
</table>

\[ g(0) = -3 \]
\[ g'(x) = \frac{24x}{(x^2 + 3)^2} \]
\[ g''(x) = \frac{24(x^2 + 3)^2 - 24x \cdot 2(x^2 + 3) \cdot 2x}{(x^2 + 3)^4} \]

Possible infl. pts.
\[ 24(x^2 + 3)^2 - 96x^2(x^2 + 3) = 0 \]
\[ 24(x^2 + 3)\left((x^2 + 3) - \frac{4}{6}x^2\right) = 0 \]
\[ (x^2 + 3) - \frac{4}{6}x^2 = 0 \]
\[ -3x^2 + 3 = 0 \]
\[ x^2 = 1 \]
\[ x = \pm 1 \]

<table>
<thead>
<tr>
<th>intervals/pt</th>
<th>(-\infty, -1)</th>
<th>-1</th>
<th>(-1, 1)</th>
<th>1</th>
<th>(1, \infty)</th>
</tr>
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<tbody>
<tr>
<td>( g''(x) )</td>
<td>neg.</td>
<td>0</td>
<td>pos</td>
<td>0</td>
<td>neg.</td>
</tr>
<tr>
<td>( g'(x) )</td>
<td>concave down</td>
<td>PoI</td>
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<td>PoI</td>
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<td></td>
<td></td>
<td></td>
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\[ s(1) = \frac{1 - 9}{1 + 3} = \frac{-8}{4} = -2 \]