2.13 The Mean Value Theorem

Rolle's Theorem

Let $a \leq b$ be real numbers, and let $f(x)$ be a function such that is

- continuous on $[a, b]$
- differentiable on $(a, b)$
- $f(a) = f(b)$

Then there exists some number $c$ in $(a, b)$ such that

$$f'(c) = 0.$$ 

(Note that $f(a) = f(b) \implies$ the average rate of change of $f$ over $[a, b]$ is 0. So, there is some point where the instantaneous rate of change equals the average rate of change.)
The Mean Value Theorem

Let $a < b$ be real numbers, and $f(x)$ a function that is:
- continuous on $[a, b]$
- differentiable on $(a, b)$.

Then there exists a number $c$ in $(a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$ 

Think: At some point (at least one), the instantaneous rate of change equals the average rate of change.
Ex: Find all values of $c$ that satisfy the conclusion of the MVT for $f(x) = e^x$ on $[1, 2]$.

Sol: MVT says: Since $f(x) = e^x$ is continuous and differentiable everywhere, there exists a number $c$ in $(1, 2)$ s.t.

$$f'(c) = \frac{f(2) - f(1)}{2 - 1}$$

$$e^c = \frac{e^2 - e}{1} = e^2 - e$$

Solve for $c$: $c = \log(e^2 - e)$. 
More interesting problems

Ex: Use Rolle's Theorem to show that \( f(x) = \sin(x) - \cos(x) \)

admits a point \( c \) between 0 and \( \frac{3\pi}{2} \) such that

\[ f'(c) = 0. \]

\[ \text{Sol}: \; f(x) = \sin(x) - \cos(x) \]

is a difference of continuous functions, hence continuous.

\[ f'(x) = \cos(x) + \sin(x) \]

exists everywhere, so \( f(x) \) is differentiable everywhere.

• Finally: \( f(0) = \sin(0) - \cos(0) = -1 \)

\[ f\left(\frac{3\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right) - \cos\left(\frac{3\pi}{2}\right) = -1 \]

So, by Rolle's Theorem, there exists a number \( c \) in

\((0, \frac{3\pi}{2})\) s.t. \( f'(c) = 0.\)
Ex: Prove that if \( f'(c) = 0 \) for all \( c \) in \([a,b]\),
then \( f(x) \) is constant on \([a,b]\).

So: It suffices to show that, for any \( c, d \) in \((a,b)\),
\( f(c) = f(d) \). Consider the subinterval \([c,d]\) inside \([a,b]\)

**Rolle's Theorem:** there exists a point \( z \) in \((c,d)\) s.t.

\[
f'(z) = \frac{f(d) - f(c)}{d - c}
\]

By assumption, \( f'(z) = 0 \) \( \Rightarrow \) \( \frac{f(d) - f(c)}{d - c} = 0 \)

\( \Rightarrow \) \( f(d) - f(c) = 0 \) \( \Rightarrow \) \( f(c) = f(d) \),

\( \Rightarrow \) \( f \) constant on \([a,b]\).
Ex: (Final 2010) If \( f(0) = 10 \) and \( f'(x) \geq 3 \) for all \( x \), what is the smallest that \( f(4) \) could possibly be?

Sol: Note that \( f'(x) \geq 3 \) for all \( x \)

\[ f'(c) = \frac{f(4) - f(0)}{4} \]

But \( f'(c) \geq 3 \) for all \( c \) between 0 and 4.

\[ \frac{f(4) - 10}{4} \geq 3 \]

\[ f(4) - 10 \geq 12 \]

\[ f(4) \geq 22 \]