3.4: Approximating functions near a specified point: Taylor polynomials

Suppose you're interested in the values of a function \( f(x) \) near a fixed point \( x = a \)
(e.g. \( f(x) = \sin(x) \) near \( x = 0 \), say \( \sin(0.1) \)).

What if \( f(x) \) is hard to evaluate?

Notice that \( y = x \) is a good approximation to \( y = \sin(x) \) for \( x \) near 0. So, for these values of \( x \), we can approximate: \( \sin(x) \approx x \).

\[ \Rightarrow \sin(0.1) \approx 0.1 \]

Motivating Questions: Given a function \( f(x) \) whose values we want to approximate for \( x \) near \( a \):

1. How do we find a function \( F(x) \) that is simple and easy to evaluate, and is a good approximation to \( f(x) \) near \( a \)?

2. What do we mean by "good approximation"? e.g. How big is the error \( |F(x) - f(x)| \)?
Our approximations: Taylor Polynomials

- named after Brook Taylor, 18th c. English mathematician

Given a function \(f(x)\), differentiable at \(x = a\),
there is a Taylor polynomial of degree \(n\), \(n \geq 0\),
that approximates \(f(x)\) on \(x\) near \(a\). This approx.
gets better as the degree \(n\) increases.

\(n = 0\): Zeroth Taylor polynomial (constant approximation)

We find a constant function \(T_0(x)\) that approximates \(f(x)\) for \(x\) near \(a\).
- Make them agree at \(x = a\):
  \[ T_0(x) = f(a) \]

Example: Approximate \(e^{0.1}\) by the
Zeroth degree Taylor polynomial to \(f(x) = e^x\) at \(x = 0\).

\[ T_0(x) = f(0) = 1. \]

So \(e^{0.1} \approx 1.\)

Notice we could do better by rotating the red line to match the slope of \(y = f(x)\) at \(x = a\).
\( n = 1 \): 1st Taylor polynomial (linear approximation / linearization)

WANT: A linear function \( T_1(x) \) that:
- agrees with \( f(x) \) at \( x = a \), i.e. \( T_1(a) = f(a) \)
- has the same slope as \( y = f(x) \) at \( x = a \):
  \( T_1'(a) = f'(a) \).

\( T_1(x) \) has the form \( c_1(x - a) + c_0 \):
- \( T_1(a) = f(a) \) \( \Rightarrow \) \( c_1(a - a) + c_0 = f(a) \)
  \( \Rightarrow \) \( c_0 = f(a) \).
- \( T_1'(a) = f'(a) \) \( \Rightarrow \) \( c_1 = f'(a) \).

\( \Rightarrow \) \( T_1(x) = f'(a)(x - a) + f(a) \)

\( T_1(x) \) is just the tangent line to \( f(x) \) at \( x = a \)!

Ex: Approximate \((5.01)^2\) using a linear approximation.

Sol: Our linear approx. is the tangent line to \( y = x^2 \)
at \( x = 5 \): \( y = 10(x - 5) + 25 \)

\( \Rightarrow (5.01)^2 \approx 10(5.01 - 5) + 25 \approx 25.1 \)