2.12 Inverse Trig Functions and their derivatives

Trigonometry you should know:

- Definition of the six basic trig functions:
  \( \sin(x), \cos(x), \tan(x), \cot(x), \csc(x), \sec(x) \)

- How to evaluate these six basic trig functions on special angles: 0, \( \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2} \) π

- and on any angle which has one of these as a reference angle.

- Special right triangles:

  \[
  \begin{align*}
  \frac{1}{2} & \quad \frac{\sqrt{3}}{2} \\
  \frac{\sqrt{2}}{2} & \quad \frac{\sqrt{2}}{2}
  \end{align*}
  \]

- How the unit circle works

- Pythagorean identities:
  \[
  \begin{align*}
  \sin^2(x) + \cos^2(x) &= 1 \\
  1 + \cot^2(x) &= \csc^2(x) \\
  \tan^2(x) + 1 &= \sec^2(x)
  \end{align*}
  \]
Inverse Trig Functions

The function \( f(x) = \sin(x) \) is not invertible.

But if we restrict its domain to \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \), then it is.

For each \( y \) in \( [-1, 1] \) there is precisely one \( x \) in \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \) such that \( \sin(x) = y \), \( \Rightarrow \) \( \sin^{-1}(y) = x \).

**Definition:** The function \( \arcsin(x) \) (or \( \sin^{-1}(x) \)) is the inverse function to \( \sin(x) \) on the interval \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \). \( \arcsin(x) \) is defined for all \( x \) in \( [-1, 1] \) and is determined by the equation \( \sin(\arcsin(x)) = x \).

**Ex:** \( \arcsin \left( \frac{1}{2} \right) = \frac{\pi}{6} \) since \( \sin \left( \frac{\pi}{6} \right) = \frac{1}{2} \).

**Ex:** \( \arcsin \left( \sin(\pi) \right) = \arcsin(0) = 0 \).

In general: \( \sin(\arcsin(x)) = x \) for all \( x \) in \( [-1, 1] \).

\( \arcsin(\sin(x)) = \) the unique angle \( \Theta \) in \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \) such that \( \sin(\Theta) = \sin(x) \).
Def: The function \( \arccos(x) \) (or \( \cos^{-1}(x) \)) is the inverse function to \( \cos(x) \) on the interval \([0, \pi]\). \( \arccos(x) \) is defined for all \( x \) in \([-1, 1]\), and is determined by the equation \( \cos(\arccos(x)) = x \).

In general, \( \cos(\arccos(x)) = x \), for all \( x \) in \([-1, 1]\).

\( \arccos(\cos(x)) \) is the unique angle \( \theta \) in \([0, \pi]\) such that \( \cos(\theta) = \cos(x) \).

Ex: \( \arccos(\cos(11\pi/6)) = \arccos(\sqrt{3}/2) = \pi/6 \).

Def: The function \( \arctan(x) \) (or \( \tan^{-1}(x) \)) is the inverse function to \( \tan(x) \) on \((-\pi/2, \pi/2)\). \( \arctan \) is defined for all real numbers, and is determined by the equation \( \tan(\arctan(x)) = x \).
For $\text{arcsec}(x)$, $\text{arccsc}(x)$, $\text{arccot}(x)$:

$\text{arcsec}(x)$

\[
\begin{align*}
  y &= \text{arcsec}(x) \\
  \sec(y) &= x \\
  x &= \frac{1}{\cos(y)} \\
  \cos(y) &= \frac{1}{x} \\
  y &= \arccos\left(\frac{1}{x}\right)
\end{align*}
\]

So $\text{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$.

Since the domain of $\arccos(x)$ is $[-1, 1]$, the domain of $\text{arcsec}(x)$ is $(-\infty, -1) \cup [1, \infty)$.

$\text{arccsc}(x)$

It turns out that $\text{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$.

Since the domain of $\arcsin(x)$ is $[-1, 1]$, the domain of $\text{arccsc}(x)$ is $(-\infty, -1] \cup [1, \infty)$.

$\text{arccot}(x)$

It turns out that $\text{arccot}(x) = \arctan\left(\frac{1}{x}\right)$.

Since domain of $\arctan(x)$ is all real numbers, the domain of $\text{arccot}(x)$ is $(-\infty, 0) \cup (0, \infty)$.

* See Definition 2.12.3 in CLP
Derivatives of Inverse Trig Functions

Ex: \( y = \arctan(x) \). Find \( \frac{dy}{dx} \).

Sol: \( y = \arctan(x) \Rightarrow \tan(y) = x \)

\[ \Rightarrow \frac{d}{dx} [\tan(y)] = \frac{d}{dx} [x] \]

\[ \Rightarrow \sec^2(y) \frac{dy}{dx} = 1 \]

So \( \frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{\sec^2(\arctan(x))} = \cos^2(\arctan(x)) \)

Let's rewrite this without trig functions by drawing a triangle.

By looking at this triangle, \( \cos^2(\arctan(x)) = \cos^2(y) \)

\[ = \left( \frac{1}{\sqrt{1+x^2}} \right)^2 \]

\[ = \frac{1}{1+x^2} \]

So: If \( y = \arctan(x) \), \( \frac{dy}{dx} = \frac{1}{1+x^2} \).

Try finding \( \frac{d}{dx} [\arcsin(x)] \) and \( \frac{d}{dx} [\arccos(x)] \) the same way!
Ex: Find $\frac{d}{dx}[\text{arcsec}(x)]$.

So: $y = \text{arcsec}(x)$. Then $y = \text{arcsin}\left(\frac{1}{x}\right)$.

Fact: $\frac{d}{dx}[\text{arcsin}(x)] = \frac{1}{\sqrt{1-x^2}}$.

To find $\frac{d}{dx}[\text{arcsin}\left(\frac{1}{x}\right)]$, use the chain rule:

$$\frac{d}{dx}[\text{arcsin}\left(\frac{1}{x}\right)] = \frac{1}{\sqrt{1-(\frac{1}{x})^2}} \cdot \frac{-1}{x^2}$$

$$= -\frac{1}{x^2\sqrt{1-(\frac{1}{x})^2}} \quad \text{or} \quad \frac{-1}{|x|\sqrt{x^2-1}}$$

**Tlm (Derivatives of Inverse Trig Functions)**

$$\frac{d}{dx}[\text{arcsin}(x)] = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx}[\text{arcsec}(x)] = \frac{-1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\text{arccos}(x)] = \frac{-1}{\sqrt{1-x^2}} \quad \frac{d}{dx}[\text{arccsc}(x)] = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\text{arctan}(x)] = \frac{1}{1+x^2} \quad \frac{d}{dx}[\text{arccot}(x)] = \frac{-1}{1+x^2}$$

**MEMORIZE**

Use the fact that $\text{arcsec}(x) = \text{arcsin}\left(\frac{1}{x}\right)$, plus the chain rule.
Chapter 3: Applications of the Derivative.

3.1: Velocity and Acceleration

the rate of change of position = velocity
the rate of change of velocity = acceleration

"the derivative of position is velocity,
the derivative of velocity is acceleration"

Ex: A particle moves in a straight line. Its position at time t is given by

\[ s(t) = t^3 - 3t^2 - 9t + 10, \]

in meters from the origin, t is measured in seconds.

On the time interval \([-2, 4]\):

Describe the particle's motion. When does it move to the left/right? What is its highest/lowest velocity? How far does it get from the origin?
\[
\text{Sol: The velocity of the particle is given by } \mathbf{v}(t) = s'(t) = 3t^2 - 6t - 9.
\]
\[
\mathbf{v}(t) = 3(t+1)(t-3)
\]
Notice: \( \mathbf{v}(t) = 0 \) when \( t = -1, t = 3 \).

So the particle is going in the same direction on the time intervals \([-\infty, -1), (-1, 3), [3, \infty]\).

So to determine the direction of travel of the particle, we test at values of \( t \) in these intervals.

\[
\mathbf{v}(-2) = 3(-1)(-5) = 15 > 0
\]

So the particle is moving to the right from \( t = -2 \) to \( t = -1 \).

\[
\mathbf{v}(1) = 3(2)(-2) < 0
\]

So moving left from \( t = -1 \) to \( t = 3 \).

\[
\mathbf{v}(4) = 3(5)(1) > 0
\]

So moving right from \( t = 3 \) to \( t = 4 \).
What about max/min velocity?

By using symmetry of parabolas, $v(2) = v(4) = 15$ is the highest velocity the particle achieves from $t=2$ to $t=4$.

The lowest velocity corresponds to the vertex of the parabola, at $t=1$: $v(1) = -12$.

To check the farthest distance from the origin the particle achieves from $t=-2$ to $t=4$, we evaluate $s(t)$ at:

- the endpoints of our time interval
- any value of $t$ where $v(t) = 0$.

$\circ s(-2) = 8$  $s(-1) = 15$

$\circ s(3) = -17$  $s(4) = -10$.

So the particle moves $15$ m away from the origin to the right and $17$ m away to the left.