0.6: Inverse Functions

Def: A function \( f(x) \) is called \textit{one-to-one} (or \textit{injective}) if \( f(x) \) never takes the same \( y \)-value more than once.

i.e. if \( x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \).

\textbf{Horizontal Line Test:} \( f(x) \) is \# one-to-one if no horizontal line passes through the graph of \( y = f(x) \) more than once.

\textbf{Ex:} \( y = x^2 \). \( y = x^3 \).
Def: Let \( f(x) \) be a one-to-one function with domain \( A \) and range \( B \). The inverse function to \( f \), denoted \( f^{-1}(x) \), has domain \( B \) and range \( A \), and is defined by

\[
\forall x \in A \quad f^{-1}(y) = x \quad \text{whenever} \quad f(x) = y
\]

\( \Rightarrow f^{-1}(f(x)) = x, \quad f(f^{-1}(x)) = x. \)

Ex: Let \( g(x) = x^3 + 2 \). What is \( g^{-1}(3) \)?
Sol: \( g(1) = 1 + 2 = 3 \); therefore \( g^{-1}(3) = 1 \).

Ex: Let \( f(x) = x^5 + 3 \). Find \( f^{-1}(x) \).
Sol: Write \( y = x^5 + 3 \). We solve for \( x \) in terms of \( y \).

\[
y - 3 = x^5 \rightarrow (y - 3)^{\frac{1}{5}} = x.
\]
So \( f^{-1}(x) = (x - 3)^{\frac{1}{5}} \).
2.10: The natural logarithm

**Def:** The natural logarithm of $x$, denoted $\log(x)$, is the inverse function to $e^x$.

$\Rightarrow \log(e^x) = x$, $e^{\log x} = x$.

**More Properties**

- $\log(e) = 1$, $\log(1) = 0$.
- $\log(xy) = \log(x) + \log(y)$
- $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$
- $\log(x^y) = y \log(x)$

**WARNING:** $(\log(x))^y \neq y \log x$

$\& \log(x+y) \neq \log(x) + \log(y)$

domain: $(0, \infty)$, range: $(-\infty, \infty)$

**Def:** $\log_a(x)$, the base-$a$ logarithm, is the inverse function to $a^x$.

$\Rightarrow \log_a(a^x) = x$, $a^{\log_a x} = x$
The derivative of $\log(x)$.

Let's try the definition of derivative:

$$\frac{d}{dx} \left[ \log(x) \right] = \lim_{h \to 0} \frac{\log(x+h) - \log(x)}{h}$$

$$= \ldots$$

$$= \lim_{h \to 0} \frac{\log \left( \frac{x+h}{x} \right)}{h}$$

This seems hard.

Let's exploit the fact that $\log(x)$ and $e^x$ are inverses:

$$e^x \cdot \frac{d}{dx} \left[ \log x \right] = 1$$

$$\log x \cdot \frac{d}{dx} \left[ \log x \right] = \frac{1}{x}$$
In general, \[
\frac{d}{dx} \left[ \log_a (x) \right] = \frac{1}{x \cdot \log(a)}
\]

Example: \[
\frac{d}{dx} \left[ \log (3x) \right].
\]

On the one hand, \[
\frac{d}{dx} \left[ \log (3x) \right] = \frac{d}{dx} \left[ \log(3) + \log(x) \right] = \frac{1}{x}
\]

On the other hand, chain rule \[
\frac{d}{dx} \left[ \log (3x) \right] = \frac{1}{3x} \cdot \frac{d}{dx} (3x) = \frac{1}{x}
\]

Example: \[
\frac{d}{dx} \left[ \log \left( \frac{1}{x^2} \right) \right]
\]
\[
= \frac{d}{dx} \left[ \log (1) - \log (x^2) \right] = \frac{d}{dx} \left[ -2 \log(x) \right] = \frac{-2}{x}
\]
Logarithmic Differentiation

Let \( y = f(x) \), and you want \( \frac{dy}{dx} \). If standard techniques don't work or are cumbersome, try taking logarithms and using the chain rule together with properties of logarithms.

Example: \( y = x^x \). Find \( \frac{dy}{dx} \).

Solution: First, take logs of both sides:

\[
\log(y) = \log(x^x)
\]

\[
\log(y) = x \cdot \log(x)
\]

Take derivatives of both sides:

\[
\frac{d}{dx} \left[ \log(y) \right] = \frac{d}{dx} \left[ x \cdot \log(x) \right]
\]

Product rule

\[
\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{x} + \log(x)
\]

So \( \frac{dy}{dx} = (1 + \log(x))y = (1 + \log(x))x^x \).