Last time:
\[
\frac{d}{dx} \sin(x) = \cos(x), \quad \frac{d}{dx} \cos(x) = -\sin(x).
\]

We can use these to obtain derivatives of the other four basic trig functions:

Ex: Let \( y = \tan(x) \). Find \( \frac{dy}{dx} \).

Sol: Recall: \( \tan(x) = \frac{\sin(x)}{\cos(x)} \).

So, \( \frac{d}{dx} \left[ \tan(x) \right] = \frac{d}{dx} \left[ \frac{\sin(x)}{\cos(x)} \right] \)

\[
= \frac{\cos^2(x) - (\sin(x))(-\sin(x))}{\cos^2(x)}
\]

\[
= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = 1 + \tan^2(x)
\]

OR \( \frac{1}{\cos^2(x)} = \sec^2(x) \).
The (list of derivatives of trig functions)
\[
\frac{d}{dx} [\sin(x)] = \cos(x) \quad \quad \frac{d}{dx} [\cos(x)] = -\sin(x)
\]
\[
\frac{d}{dx} [\tan(x)] = \sec^2(x) \quad \quad \frac{d}{dx} [\sec(x)] = \sec(x)\tan(x)
\]
\[
\frac{d}{dx} [\cot(x)] = -\csc^2(x) \quad \quad \frac{d}{dx} [\csc(x)] = -\csc(x)\cot(x)
\]

Exercise: Show that \(\frac{d}{dx} [\cot(x)] = -\csc^2(x)\) by writing \(\cot(x) = \frac{\cos(x)}{\sin(x)}\) and using the quotient rule.

Ex: Differentiate \(f(x) = (e^x + \cot(x))(5x^6 - \csc(x))\).

So: \(f(x) = g(x)h(x)\), where
\[
g(x) = e^x + \cot(x) \rightarrow g'(x) = e^x - \csc^2(x)
\]
\[
h(x) = 5x^6 - \csc(x) \rightarrow h'(x) = 30x^5 + \csc(x)\cot(x)
\]
\[
f'(x) = g(x)h'(x) + g'(x)h(x)
\]
\[
= (e^x + \cot(x))(30x^5 + \csc(x)\cot(x))
\]
\[
\quad + (e^x - \csc^2(x))(5x^6 - \csc(x))
\]
Ex: Let \( f(x) = \begin{cases} x^2 \cos \left( \frac{1}{x} \right), & x \neq 0 \\ 0, & x = 0 \end{cases} \).

Is \( f(x) \) differentiable at \( x = 0 \)?

So: Does \( f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \) exist?

\[
\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \cos \left( \frac{1}{x} \right)}{x} = \lim_{x \to 0} x \cos \left( \frac{1}{x} \right).
\]

We show this limit exists by using the squeeze theorem.

\[-1 \leq \cos \left( \frac{1}{x} \right) \leq 1\]

\[-x \leq x \cos \left( \frac{1}{x} \right) \leq x\]

Since \( \lim_{x \to 0} -x = \lim_{x \to 0} x = 0 \), \( \lim_{x \to 0} x \cos \left( \frac{1}{x} \right) = 0 \) by the squeeze theorem.

So, yes, \( f'(0) \) exists.
2.9: The **Chain Rule**: derivatives of composite functions

**Ex**: The position of a mountain climber at time $t$ is given by $h(t)$. The air temperature $T$ meters above sea level is given by $T(h)$.

$\therefore$ temperature of the mountain climber at time $t$ is $T(h(t))$, with respect to $t$.

What is the rate of change of the temperature the mountain climber feels?

*Common sense:* $T'(h(t))$ should be negative.

**Notice**: $h'(t) > 0$, $T'(h) < 0$.

So, the sign of $T'(h(t))$ matches that of $h'(t) \cdot T'(h)$.

**Chain Rule**: Let $f(x)$, $g(x)$ be differentiable functions. Then:

$$\frac{d}{dx} \left[ f(g(x)) \right] = f'(g(x)) \cdot g'(x)$$
Chain Rule - Alternate Version

Let \( y = f(u) \) and \( u = g(x) \). Then

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.
\]

Ex: Compute \( \frac{d}{dx} \left( (\sin(x))^2 \right) \)

So, \( f(u) = u^2 \) \( \Rightarrow \) So if \( y = f(u) \),
\( u = g(x) = \sin(x) \) \( \Rightarrow \) Then \( y = (\sin(x))^2 \).

Chain rule says:
\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.
\]

\[
\frac{dy}{du} = \frac{d}{du} (u^2) = 2u, \quad \frac{du}{dx} = \frac{d}{dx} [\sin(x)] = \cos(x).
\]

So:
\[
\frac{dy}{dx} = 2 \sin(x) \cdot \cos(x).
\]
Ex: Compute \( \frac{d}{dz} \left( (3z+1)^{10} \right) \). \( = 10(3z+1)^9 \cdot 3 \)

Sol: Let \( y = f(u) = u^{10} \), \( u = g(z) = 3z+1 \).

Chain rule: \( \frac{dy}{dz} = \frac{dy}{du} \cdot \frac{du}{dz} \).

\[
\frac{dy}{du} = 10u^9 \quad \frac{du}{dz} = 3.
\]

So \( \frac{dy}{dz} = 30u^9 = 30(3z+1)^9 \).

\[
= 10 \cdot \frac{(3z+1)^9 \cdot 3}{f(u) \cdot g'(z)}.
\]
Ex: \( \frac{d}{dx} \left[ \tan(\sqrt{x}) \right] \) \( = \sec^2(\sqrt{x}) \cdot \frac{1}{2} x^{-\frac{1}{2}} \)

So: Let \( y = f(u) = \tan(u) \)
\( u = g(x) = \sqrt{x} \)
\[ y = \tan(\sqrt{x}) \]

\[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \]
\[ \frac{dy}{du} = \sec^2(u), \quad \frac{du}{dx} = \frac{1}{2} x^{-\frac{1}{2}} \]
\[ = \sec^2(\sqrt{x}) \]

So \( \frac{dy}{dx} = \sec^2(\sqrt{x}) \cdot \frac{1}{2} x^{-\frac{1}{2}} \).
Ex: \[
\frac{d}{dx} \left[ u \sqrt{\sin(e^x)} \right] = \frac{1}{4} (\sin(e^x))^{-3/4} \cdot \cos(e^x) \cdot e^x
\]

So: Let \( y = f(u) = u^{1/4} \)

\( u = g(x) = \sin(e^x) \)

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{4} u^{-3/4} \cdot \frac{du}{dx}
\]

\[
\frac{du}{dx} = \frac{du}{dw} \cdot \frac{dw}{dx}
\]

\( u = g(w) = \sin(w) \)

\( w = h(x) = e^x \)

So: \( \frac{du}{dx} = \cos(w) \cdot e^x \)

So: \( \frac{dx}{du} = \frac{1}{4} (\sin(e^x))^{-3/4} \cdot \cos(e^x) \cdot e^x \)

\[
\left[ \text{Notice:} \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dw} \cdot \frac{dw}{dx} \right]
\]
Ex: The radius of a circular oil spill expands at a rate of $2 \text{ m/hr}$. How fast is the area of the circle of oil expanding 1 hour after the spill?

So: $A = \pi r^2$, $A(t) = \pi (r(t))^2$

Want to know: $A'(t)$ at $t=1$.

Well, $A'(t) = \pi \cdot 2r(t) \cdot r'(t)$

$A'(1) = \pi \cdot 2r(1) \cdot r'(1)$

$r(1) = 2$, $r'(1) = 2$

$A'(1) = 8\pi$.

So the oil spill has area $8\pi \text{ m}^2$, 1 hour after the spill occurs.
Ex: Suppose \( f(x) \) is differentiable at \( x = 1 \).
\[ f(1) = 2, \quad f'(1) = 5. \]
Let \( g(x) = x^2 \cdot f(x) \). Find the equation of the tangent line to \( y = g(x) \) at \( x = 1 \).

So: The equation of the tangent line to \( y = g(x) \) at \( x = 1 \) is:
\[ y = g'(1)(x-1) + g(1) \]
\[ g(1) = 2 \]
\[ g'(x) = x^2 \cdot f'(x) + f(x) \cdot 2x \]
\[ g'(1) = 5 + 4 = 9. \]
So \( y = 9(x-1) + 2 \).
Ex: The position of a particle at time $t$ is given by $s(t) = \frac{1}{3} t^3 - 4t^2 + 15t$.

Find the total distance traveled by the particle in the first 10 seconds.

So: The particle changes direction when $v(t) = 0$.

$v(t) = t^2 - 8t + 15$

$= (t-3)(t-5)$

$\Rightarrow v(t) = 0$

when $t = 3$ and $t = 5$.

So the particle is moving in the same direction for all $t$ in $(0, 3)$, and for all $t$ in $(3, 5)$, and for all $t$ in $(5, 10)$.

Total distance traveled, then, is:

$|s(3) - s(0)| + |s(5) - s(3)| + |s(10) - s(5)|$.

Plug in, evaluate, done.