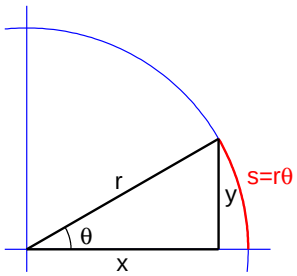
UBC MATH 255 FORMULAS FOR MECH 221

BASIC TRIGONOMETRY



$$\cos\theta = \frac{x}{r} = \frac{1}{\sec\theta}$$
$$\sin\theta = \frac{y}{r} = \frac{1}{\csc\theta}$$
$$\tan\theta = \frac{y}{x} = \frac{1}{\cot\theta}$$
$$\cos^2\theta + \sin^2\theta = 1$$
$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$
$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

$$\sin(0) = 0 = \cos\left(\frac{\pi}{2}\right)$$
$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$$
$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \cos\left(\frac{\pi}{4}\right)$$
$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right)$$
$$\sin\left(\frac{\pi}{2}\right) = 1 = \cos(0)$$

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$
$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

QUADRATIC FORMULA (valid when $a \neq 0$)

$$az^2 + bz + c = 0 \iff z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

DERIVATIVES

$$(fg)' = f'g + fg'$$
$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$
$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$
$$\frac{d}{dx}\sin(\omega x) = \omega\cos(\omega x)$$
$$\frac{d}{dx}\cos(\omega x) = -\omega\sin(\omega x)$$
$$\frac{d}{dx}\tan(\omega x) = \frac{\omega}{\cos^2(\omega x)}$$
$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$
$$\frac{d}{dx}e^{sx} = se^{sx}$$
$$\frac{d}{dx}\ln|kx| = \frac{1}{x}$$
$$\frac{d}{dx}x^p = px^{p-1}$$

HARMONIC MOTION ($\omega > 0$ constant): THREE EQUIVALENT FORMS

Form AB:	$u(t) = A\cos(\omega t) + B\sin(\omega t)$		$A \in \mathbb{R},\ B \in \mathbb{R}$
Form C:	$u(t) = C\cos(\omega t + \phi) = C\sin(\omega t + [\phi + \frac{\pi}{2}])$		$\phi \in \mathbb{R},\ C > 0$
Form D:	$u(t) = \mathrm{Re}\left\{De^{i\omega t}\right\}$		$\phi \in \mathbb{R},\ D \in \mathbb{C}$
Conversion Helpers:	$A = C\cos\phi$	$B = -C\sin\phi$	$D = A - iB = Ce^{i\phi}$
Euler's Formula:	$e^{i\theta} = \cos\theta + i\sin\theta$	$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$	$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

NEWTON'S ITERATION (for solving $g(z) = 0$)

$$z_{n+1} = z_n - \frac{g(z_n)}{g'(z_n)}$$

LINEAR ODE's: SCALAR UNKNOWN $y = y(t)$, CONSTANT COEFFICIENTS

Differential Operator: $L[y] = a_0 y + a_1 \dot{y} + a_2 \ddot{y} + \ldots + a_{n-1} y^{(n-1)} + a_n y^{(n)}, \qquad a_n \neq 0$

Characteristic Polynomial: $p(s) \stackrel{\text{def}}{=} a_0 + a_1 s + a_2 s^2 + \ldots + a_{n-1} s^{n-1} + a_n s^n$

Homogeneous ODE: If s is a zero for p, i.e., if $p(s) = 0$, then one solution of $L[y] = 0$ is $y = e^{st}$.

Repeated Roots: If s is a zero of multiplicity k for p, then k solutions of $L[y] = 0$ are e^{st}, te^{st}, ..., $t^{k-1}e^{st}$.

General Solution for $L[y] = 0$: $y_h = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n,$ where $y_1, \ldots, y_n$ are independent solutions of $L[y] = 0$

General Solution for $L[y] = f$: $y = y_h + y_p,$ with y_h as above, where y_p is any particular solution of $L[y] = f$

Exponential Shift: $L[e^{st}u(t)] = e^{st}\left(p(s)u(t) + \frac{p'(s)}{1!}\dot{u}(t) + \frac{p''(s)}{2!}\ddot{u}(t) + \ldots + \frac{p^{(n)}(s)}{n!}u^{(n)}(t)\right)$

LINEAR ODE's: SCALAR UNKNOWN y, VARIABLE COEFFICIENTS, FIRST ORDER ONLY

$$y'(t) = a(t)y(t) + f(t),\ y(t_0) = y_0 \quad \Longleftrightarrow \quad y(t) = e^{A(t)}\left[y_0 + \int_{t_0}^{t} e^{-A(r)} f(r)\, dr\right], \quad \text{where} \quad A(t) = \int_{t_0}^{t} a(u)\, du$$

LINEAR ODE's: VECTOR UNKNOWN $\mathbf{x}$, CONSTANT COEFFICIENTS IN MATRIX $\mathbf{A}$, FIRST ORDER ONLY

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t),\ \mathbf{x}(t_0) = \mathbf{x}_0 \quad \Longleftrightarrow \quad \mathbf{x}(t) = e^{(t-t_0)\mathbf{A}}\mathbf{x}_0 + \int_{t_0}^{t} e^{(t-r)\mathbf{A}}\mathbf{f}(r)\, dr$$

$$e^{t\mathbf{A}} = \mathbf{I} + \frac{(t\mathbf{A})}{1!} + \frac{(t\mathbf{A})^2}{2!} + \frac{(t\mathbf{A})^3}{3!} + \cdots$$

$e^{t\mathbf{A}} = \mathbf{X}(t)\mathbf{X}(0)^{-1}$ for any invertible matrix solution $\mathbf{X}$ for $\dfrac{d\mathbf{X}}{dt} = \mathbf{A}\mathbf{X}$

$\mathbf{x}(t) = e^{st}\mathbf{v}$ satisfies $\mathbf{x}' = \mathbf{A}\mathbf{x}$ whenever $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ with eigenvalue s

$\mathbf{x}(t) = e^{st}(t\mathbf{v} - \mathbf{w})$ satisfies $\mathbf{x}' = \mathbf{A}\mathbf{x}$ whenever $\mathbf{v}$ is an eigenvector of A with eigenvalue s and $(s\mathbf{I} - \mathbf{A})\mathbf{w} = \mathbf{v}$

TAYLOR AND MACLAURIN SERIES

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \pm \cdots, \qquad z \in \mathbb{C} \qquad\qquad e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots, \qquad z \in \mathbb{C}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} \pm \cdots, \qquad z \in \mathbb{C} \qquad\qquad \frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots, \qquad |z| < 1$$

$$P_n(x) = f(a) + f'(a)[x-a] + \frac{f''(a)}{2!}[x-a]^2 + \cdots + \frac{f^{(n)}(a)}{n!}[x-a]^n$$

$f(x) = P_n(x) + \dfrac{f^{(n+1)}(\xi)}{(n+1)!}[x-a]^{n+1}$ for some ξ between a and x

NUMERICAL APPROXIMATIONS FOR THE INTEGRAL $I = \int_a^b f(x)\, dx$

Equal spacing with N subintervals: $h = \dfrac{b-a}{N}$ Notation: $x_k = a + kh,\ f_k = f(x_k),\ k = 0, 1, \ldots, N$

Trapezoidal: $T_N = \dfrac{h}{2}\left[f_0 + 2f_1 + \ldots + 2f_{N-1} + f_N\right] = I + \dfrac{f''(\xi)}{12}[b-a]h^2$ (for some ξ between a and b)

Simpson's (needs N even): $S_N = \dfrac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \ldots + 2f_{N-2} + 4f_{N-1} + f_N\right] = I + \dfrac{f^{(4)}(\xi)}{180}[b-a]h^4$

NUMERICAL APPROXIMATIONS FOR DERIVATIVES ($h > 0$)

$$f'(a) \approx \frac{1}{h}\left[f(a) - f(a-h)\right] \qquad f'(a) \approx \frac{1}{2h}\left[f(a+h) - f(a-h)\right] \qquad f'(a) \approx \frac{1}{h}\left[f(a+h) - f(a)\right]$$

$$f''(a) \approx \frac{1}{h^2}\left[f(a+h) - 2f(a) + f(a-h)\right]$$

LAPLACE TRANSFORMS

$g = \mathcal{L}^{-1}\{G\}$	$G = \mathcal{L}\{g\}$	Notes
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)\,dt$	Definition.
e^{ct}	$\dfrac{1}{s-c}$	Works for each complex c.
t^p	$\dfrac{p!}{s^{p+1}}$	Works for each $p = 0, 1, 2, \ldots$.
$\sin(\omega t)$	$\dfrac{\omega}{s^2+\omega^2}$	
$\cos(\omega t)$	$\dfrac{s}{s^2+\omega^2}$	
$\sin^2(\omega t)$	$\dfrac{2\omega^2}{s(s^2+4\omega^2)}$	
$\cos^2(\omega t)$	$\dfrac{s^2+2\omega^2}{s(s^2+4\omega^2)}$	
$t\sin(\omega t)$	$\dfrac{2\omega s}{(s^2+\omega^2)^2}$	
$\sin(\omega t) - \omega t\cos(\omega t)$	$\dfrac{2\omega^3}{(s^2+\omega^2)^2}$	
$y'(t)$	$sY(s) - y(0)$	
$y''(t)$	$s^2Y(s) - sy(0) - y'(0)$	
$e^{ct}f(t)$	$F(s-c)$	Shift Theorem #1.
$u(t-c)f(t-c)$	$e^{-cs}F(s)$	Shift Theorem #2.
$(f \star g)(t)$	$F(s)G(s)$	$(f \star g)(t) \stackrel{\text{def}}{=} \int_0^t f(t-r)g(r)\,dr$
$\delta(t-c)f(t)$	$e^{-cs}f(c)$	Assumes f continuous at c.

Alternate Notation: $u_c(t) = u(t-c), \quad \delta_c(t) = \delta(t-c), \qquad c \in \mathbb{R}$.

INDEFINITE INTEGRALS

$$\int x\sin(bx)\,dx = \frac{\sin(bx)}{b^2} - \frac{x\cos(bx)}{b} \qquad \int x\cos(bx)\,dx = \frac{\cos(bx)}{b^2} + \frac{x\sin(bx)}{b}$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1) \qquad \int \tan x\,dx = \ln|\sec x|$$

$$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} \qquad \int e^{ax}\cos(bx)\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x \qquad \int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x$$

$$\int \sec^2 x\,dx = \tan x \qquad \int \tan^2 x\,dx = \tan x - x$$

$$\int (a^2-x^2)^{3/2}\,dx = \frac{x}{8}(5a^2-2x^2)\sqrt{a^2-x^2} + \frac{3a^4}{8}\sin^{-1}\frac{x}{a} \qquad \int (x^2\pm a^2)^{3/2}\,dx = \frac{x}{8}(2x^2\pm 5a^2)\sqrt{x^2\pm a^2} + \frac{3a^4}{8}\ln\left|x+\sqrt{x^2\pm a^2}\right|$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) \qquad \int \sqrt{x^2\pm a^2}\,dx = \frac{x}{2}\sqrt{x^2\pm a^2} \pm \frac{a^2}{2}\ln\left|x+\sqrt{x^2\pm a^2}\right|$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| \quad (a>0) \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} \quad (a>0)$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) \quad (a>0) \qquad \int \frac{dx}{\sqrt{x^2\pm a^2}} = \ln\left|x+\sqrt{x^2\pm a^2}\right|$$

$$\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2-x^2}} \qquad \int \frac{dx}{(x^2\pm a^2)^{3/2}} = \frac{\pm x}{a^2\sqrt{x^2\pm a^2}}$$