

MECH 221 MATH LEARNING GUIDE — WEEK TEN (starts 2014-11-24)

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Lecture Schedule.

- 2014-11-24 (Mon): MATH 24, Methods of Euler and Heun for ODE IVP's
2014-11-26 (Wed): MATH 25, Approximate Integration
2014-11-28 (Fri): MATH 26, Error Estimates and Richardson Extrapolation

Overview. We make a complete conceptual shift to the practical job of using a computer's immense appetite for calculation to produce numerical approximations for calculus-related problems. Differential equations are, of course, our focus, but ideas born in that study can be used also to find ordinary definite integrals and derivatives.

Learning Goals. You have mastered this week's material when you can ...

1. Derive and apply Euler's Method for approximating the solution of an initial-value problem. Remember and use the fact that an N -step calculation with uniform mesh width h will produce an approximation whose error is $O(h)$.
2. Derive and apply Heun's (Improved Euler) Method for approximating the solution of an initial-value problem. Remember and use the fact that an N -step calculation with uniform mesh width h will produce an approximation whose error is $O(h^2)$.
3. Define and evaluate the average value of a given function $f(x)$ on a given interval $[a, b]$. This is the constant \bar{f} such that

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x) dx.$$

4. Calculate an approximate value for a definite integral using the Trapezoidal Rule. Start with either a table of function-values or a description of the evaluation-points and a formula for the integrand. Remember and use the fact that a calculation with uniform mesh width h will produce an approximation whose error is $O(h^2)$.
5. Calculate an approximate value for a definite integral using Simpson's Rule. Start with either a table of function-values or a description of the evaluation-points and a formula for the integrand. Remember and use the fact that a calculation with uniform mesh width h will produce an approximation whose error is $O(h^4)$.
6. [Richardson Extrapolation] Find the constants A and B in an approximation of the form

$$f(h) \approx A + Bh^\gamma \quad \text{for } h \approx 0, \quad (*)$$

where the exponent γ is given (or remembered), along with numerical values for $f(h_1)$ and $f(h_2)$ at known points h_1 and h_2 . Use the result to predict $\lim_{h \rightarrow 0} f(h)$.

7. Given more than two computed values for $f(h)$, build an approximation like (*) by selecting the integer exponent γ and the constants A and B .
8. Apply Richardson Extrapolation to discrete approximations for derivatives and integrals to produce higher-order approximations.

Textbook Sections.

- **JL 1.7 — Numerical methods: Euler's method:** Try problems 1.7.3–6 (use a calculator), then 1.7.101–103 (use a spreadsheet).
- **WFT 3.1 — Euler's Method:** Focus on the first half of this section, where the plain Euler method is described. The second half describes a "semilinear Euler method", which we did not study. Try a few problems from #1–12 on a spreadsheet, then make sure you can do small-scale ones like #1–6 on your calculator.

- **WFT 3.2 — The Improved Euler and Related Methods:** Trench says “Improved Euler” for the method we attribute to Heun; this book attaches Heun’s name to another method that it is safe to ignore. Power through a few samples from #1–5 using a spreadsheet.
- **WFT 3.3 — The Runge-Kutta Method:** FYI, this method is the foundation for Matlab’s `ode45` function. But that’s as close to the details as we are going to get. If you are curious, read this section to satisfy your personal interest. It will not be tested.
- **Wetton’s Notes 4.3 — Richardson Extrapolation:** We saw this in class, except that we used the letter γ for the exponent that Wetton calls p . Our choice is based on observing that p is already heavily used for other meanings in this course. Try Exercises 14 and 15 (page 26).