

MECH 221 MATH LEARNING GUIDE — WEEK SEVEN (starts 2014-11-03)
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Lecture Schedule.

- 2014-11-03 (Mon): MATH 16, Matrix Exponentials
- 2014-11-04 (Tue): MATH 17, Laplace Transform – Introduction
- 2014-11-05 (Wed): MATH 18, Laplace Transform – Properties and Calculations

Overview. This week we introduce the Laplace Transform, a new approach to familiar problems that will open the door to new ideas next week.

Learning Goals. You have mastered this week’s material when you can . . .

1. Construct the matrix exponential e^{At} for any given 2×2 matrix A , and for any nondefective 3×3 matrix A , by each of the methods below:

□ Solving for each column separately, recalling that column k is the unique solution $\vec{\phi}^{(k)}$ of the IVP

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x}(0) = \hat{\mathbf{e}}^{(k)}.$$

□ Reducing to a higher-order equation for a scalar unknown, solving that, and reassembling the components for the column-vector solutions mentioned above.

□ Building an invertible “fundamental solution” matrix $X(t)$ from linearly independent solution columns, and then taking $e^{At} = X(t)X(0)^{-1}$.

2. Remember/recognize and exploit the theoretical properties of $\Phi(t) \stackrel{\text{def}}{=} e^{At}$, such as $\Phi(0) = I$, $\Phi(t)^{-1} = \Phi(-t)$, $\Phi(s+t) = \Phi(s)\Phi(t)$, and $\dot{\Phi}(t) = A\Phi(t) = \Phi(t)A$.
3. Recognize and apply Duhamel’s solution formula for the nonhomogeneous IVP, namely,

$$\left[\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{g}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \right] \iff \mathbf{x}(t) = e^{At}\mathbf{x}_0 + e^{At} \int_0^t e^{-Ar} \mathbf{g}(r) dr.$$

4. Calculate the Laplace Transform of a given function directly from the definition,

$$F = \mathcal{L}\{f\} \iff F(s) = \int_0^\infty e^{-st} f(t) dt.$$

Evaluate the right side correctly even when the input function $f(t)$ has a piecewise definition.

5. Find the Laplace Transform of any polynomial function (including a constant).
6. Find the Laplace Transform of any sinusoidal function.
7. Find the Laplace Transform of $f(t) = q(t)e^{ct}$, where q is a polynomial and c is a real constant.
8. Find the Laplace Transform of $f(t) = q(t)e^{ct}$, where q is a polynomial and c is a *complex* constant. Use Euler’s formula to deduce the transforms for $q(t) \sin(\omega t)$ and $q(t) \cos(\omega t)$.
9. Recognize and use Shift Theorem #1 in the general form below, for any complex constant c :

$$\mathcal{L}\{e^{ct} f(t)\} = \mathcal{L}\{f(t)\}_{s \rightarrow (s-c)} = F(s-c).$$

10. Use the Laplace Transform to simplify a given constant-coefficient linear differential equation or system, and isolate the Laplace Transform of the unknown function.

11. Use the following methods of “aggressive algebra” to rewrite a given Laplace Transform as a sum of terms that are each easy to invert:
- add+subtract trick,
 - multiply/divide trick,
 - partial fractions (including cases with repeated real or complex roots).
12. Given $Y(s)$, find $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ using the methods described above. In particular, apply Shift Theorem #1 in the form

$$\mathcal{L}^{-1}\{F(s-c)\} = e^{ct}\mathcal{L}^{-1}\{F(s)\} = e^{ct}f(t).$$

13. Determine the stability of a given Laplace Transform $Y(s)$ by analyzing the locations of its poles in the complex plane (“the s -plane”).

Textbook Sections.

- JL 3.8 — Matrix exponentials:** Prof Lebl gives an interesting alternative construction of the matrix exponential based on power series. Learn it if you like, but it’s optional. Using whatever method you prefer, make sure you can solve 3.8.3, 3.8.5, 3.8.8, 3.8.10, 3.8.101–3.8.103.
- JL 3.9 — Nonhomogeneous systems:** We focus on constant-coefficient systems of first order, so we can skip sections 3.9.2 and 3.9.3. In Section 3.9.1, our preferred approach (exponential shift) is very similar to what this text calls the “integrating factor” method. Focus on this part of subsection 3.9.1. The next part, Eigenvector decomposition, is interesting but not necessary. The final part, Undetermined coefficients, was not mentioned in class but makes a rewarding topic for self-study. Just make sure you have at least one working method for solving 3.9.4–3.9.7 and 3.9.101–3.9.103.
- JL 6.1 — The Laplace transform:** Solve all of 6.1.5–6.1.15 and 6.1.101–6.1.104.
- JL 6.2 — Transforms of derivatives and ODEs:** This material will contribute to lectures both this week and next week. For now, try 6.2.3–6.2.5.
- WFT 10.7 — Variation of Parameters for Nonhomogeneous Linear Systems:** We don’t like “Variation of Parameters”, because we have our own methods for nonhomogeneous linear systems. Use exponential shift or MUC to solve some problems from 1–10. Then, noting that whenever $Y(t)$ is a “fundamental solution” for $\dot{\mathbf{x}} = A\mathbf{x}$ one has $e^{At} = Y(t)Y(0)^{-1}$, solve a few examples from 11–16.
- WFT 8.1 — Introduction to the Laplace Transform:** Solve #1, 2, 4, 5, 7, 12.
- WFT 8.2 — The Inverse Laplace Transform:** Solve #1–8.
- WFT 8.3 — Solution of Initial Value Problems:** Practice on some selections from #1–37.

Next Week’s Test. On Friday 14 November 2014, there will be a 50-minute test starting at 08:00. Out of the 35 marks available in total, the number designated for “Math” is 12.5. (Expect a 25-mark problem that combines Math with Dynamics.)