

MECH 221 MATH LEARNING GUIDE — WEEK SIX (starts 2014-10-27)
UBC MECH 2 Materials ©2014 by Philip D. Loewen

Lecture Schedule.

- 2014-10-27 (Mon): MATH 13, Linear Systems with Real Eigenvalues
2014-10-28 (Tue): MATH 14, Linear Systems with Complex Eigenvalues; Phase Portraits
2014-10-29 (Wed): MATH 15, Defective Matrices and Matrix Exponentials

Overview. This week we focus on systems of first-order linear ODE's with constant coefficients.

Learning Goals. You have mastered this week's material when you can ...

1. Draw the nullclines and slope field in the phase plane for the homogeneous system of equations

$$\dot{\mathbf{x}}(t) = A\mathbf{x}, \quad (*)$$

in any situation involving a constant 2×2 matrix A .

2. Use eigenvalue and eigenvector information as appropriate to add some approximate system trajectories to complete the phase plane portrait begun in Goal 2. Given a 2×2 system, classify the origin as a *node*, *saddle point*, *spiral point*, or *centre*.
3. Find a vector-valued formula for the general solution of (*) for any situation constant 2×2 matrix A . Be prepared to treat matrices with real eigenvalues, complex eigenvalues, and/or repeated eigenvalues with a defective eigenspace.
4. Find a vector-valued formula for the solution of any initial-value problem of size 2×2 , with the form

$$\dot{\mathbf{x}}(t) = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0.$$

5. Translate any scalar equation of the form $ay' + by + cy = g(t)$ into the vector-matrix form $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{G}(t)$ by making suitable definitions for x_1 , x_2 , A , and \mathbf{G} . (Usually $x_1 = y$ and $x_2 = y'$.)
6. Given a solution for a vector-matrix system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{G}(t)$ that originates from the scalar case as in Goal 3, find the solution $y(t)$ for the original problem.
7. Given a vector-matrix system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{G}(t)$ where A has size 2×2 , derive a second-order ODE for one of the component functions $x_1(t)$ or $x_2(t)$, solve that equation by known methods, and use the result to find the vector-valued solution $\mathbf{x}(t)$.
8. Recognize and correctly apply the terms *stable* and *unstable* to a given system of the form $\dot{\mathbf{x}} = A\mathbf{x}$.
9. Repeat Goals 3–8 for first-order systems of size $n \times n$, or for scalar ODE's involving derivatives of order up to n , for various $n \geq 3$. Know the general structure and notation for arbitrary n ; provide full details in situations where either (i) computer assistance is available, as it is for your homework, or (ii) some extra hint is given to help find $n - 2$ out of the n eigenvalues you will need.

Textbook Sections.

- **JL 3.1 — Introduction to systems of ODEs:** Read the whole story, try all exercises in 3.1.1.
- **JL 3.2 — Matrices and linear systems:** Try the exercises in 3.2.6 first. This section is all about prerequisites for Mech 2. Read it if you don't remember the material; skip it if you do.
- **JL 3.3 — Linear systems of ODEs:** Browse through this, and try 3.3.1, 3.3.103, 3.3.104. (We didn't give much class time to the concept of linear independence; don't start now.)
- **JL 3.4 — Eigenvalue method:** Try 3.4.5–3.4.11 and 3.4.101–3.4.104.
- **JL 3.5 — Two dimensional systems and their vector fields:** Here we see a variety of intuition-building pictures. Among the exercises, 3.5.3 and 3.5.103 involve coefficient matrices with an eigenvalue of 0. Pay special attention to these, because if you can stretch your knowledge to explain them, you probably have a robust understanding of the key concepts. But make sure you can do all six problems in 3.5.1.

- **JL 3.6 — Second order systems:** Skip this.
- **JL 3.7 — Multiple eigenvalues:** The opening paragraph is worth knowing, and you can see its philosophy at work in the dirty hack that comes up in Computer Lab 5. In class, we used exponential shift to overcome the problem of a defective eigenvalue. Try 3.7.2–3.7.4.
- **WFT 10.4 — Constant Coefficient Homogeneous Linear Systems I:** Try a few of the IVP’s, 10.4 #16–27. Start a few of the phase plane plots in #29–34 by hand, then finish by computer. Read #35 carefully and formulate your own explanations for the statements it makes.
- **WFT 10.5 — Constant Coefficient Homogeneous Systems II:** This is more reading than the defective case deserves. If you can solve problems 1–4 and 13–16, it’s enough.
- **WFT 10.6 — Constant Coefficient Homogeneous Systems III:** The case of complex eigenvalues is important, so do pay attention to this section. However, Trench’s detailed discussion of “shadow trajectories” starting on page 563 goes into more detail than we will ever need. Practice finding some general solutions (#1–6), solving some IVP’s (#17–22), and drawing some pictures (#29–34). For problems 29–34, just use the nullclines and the real part of the eigenvalues to make a reasonable approximate sketch—ignore all mention of the shadow vectors \mathbf{U} and \mathbf{V} .

Next Week’s Test. On Thursday 06 November 2014, there will be a 110-minute test starting at 08:00. Out of the 75 marks available in total, the number designated for “Math” is 0.