

# MECH 221 MATH LEARNING GUIDE — WEEK FIVE (starts 2014-10-20)

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## Lecture Schedule.

2014-10-20 (Mon):	MATH 09, Transfer Functions, and an RLC circuit
2014-10-21 (Tue):	MATH 10, All our Math Methods on an RC circuit
2014-10-22 (Wed):	MATH 11, First-order ODE's—separable, or linear time-varying
2014-10-24 (Fri):	MATH 12, Systems of first-order ODE's, Linear systems

**Overview.** This is a big week. We wrap up our discussion of scalar linear ODE's with constant coefficients with a discussion of frequency response inspired by circuits. Then we take a quick side-trip to meet the simplest possible nonlinear ODE's and to note that, even in the linear case, new methods will be needed if the coefficients are not constant. Finally, we consider systems of ODE's, focus first on the constant-coefficient linear case.

**Learning Goals.** You have mastered this week's material when you can . . .

1. Write the transfer function (from  $u$  to  $y$ ) for a given differential relationship of the form

$$a_n y^{(n)} + \dots + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = b_m u^{(m)} + \dots + b_1 \dot{u} + b_0 u.$$

2. Determine, as functions of  $\omega > 0$ , the amplitude  $C = C(\omega)$  and phase  $\phi = \phi(\omega)$  of the steady-state response  $y(t) = C \cos(\omega t + \phi)$  that results when the input  $u(t) = \cos(\omega t)$  is applied to the system above.
3. Recognize when a given ODE is *separable*. Express solution curves for a separable ODE as implicit relations like  $F(y) = G(x)$ .
4. Recognize when it is possible to rearrange a given implicit relation like  $F(y) = G(x)$  to present  $y$  as an explicit function of  $x$ ; actually do this rearrangement in such cases.
5. Find the general solution for the homogeneous linear ODE  $y' + p(t)y = 0$ , in cases where the coefficient  $p(t)$  is not constant.
6. Find the general solution for an inhomogeneous linear ODE of the form  $y' + p(t)y = g(t)$ , even in cases where the coefficient  $p(t)$  is not constant.
7. Find the unique solution to an initial-value problem involving an ODE of the form  $y' + p(t)y = g(t)$ .
8. Use careful reading, physical principles, and common sense to invent differential equations that describe mixing problems, fluid flow problems, temperature-change problems, etc.
9. Translate a scalar equation of the form  $ax'' + by' + cy = g(t)$  into the vector-matrix form  $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{G}(t)$  by making suitable definitions for  $x_1, x_2, A$ , and  $\mathbf{G}$ .
10. Apply the basic fact that a nonzero function  $\mathbf{x}(t) = e^{st}\mathbf{v}$  satisfies  $\dot{\mathbf{x}} = A\mathbf{x}$  if and only if  $s$  is an eigenvalue of  $A$  with eigenvector  $\mathbf{v}$ .
11. [Prereq review] Given a nonzero vector  $\mathbf{v}$  in  $\mathbb{R}^n$  and an  $n \times n$  matrix  $A$ , determine whether  $\mathbf{v}$  is an eigenvector for  $A$ , and, if so, find the corresponding eigenvalue.
12. [Prereq review] Compute the characteristic polynomial  $p(s) = \det(sI - A)$  of a given square matrix  $A$ . Know enough about expanding determinants to be able to do this for a matrix of any size.
13. [Prereq review] Find all eigenvalues for a given matrix  $A$  of size  $2 \times 2$ , and determine all eigenvectors corresponding to each *real* eigenvalue.

## Textbook Sections.

- **JL 2.6 — Forced oscillations and resonance:** The written description captures the intuitive meaning and importance of our work on frequency response and transfer functions. Please figure out what the graph in Figure 2.8 (page 82) is showing.

- **JL 6.2.4 — Transfer functions:** On this single page, you can see transfer functions in their natural setting, namely, the world of Laplace Transforms. We'll study that topic later this term. For now, focus on Example 6.2.3, and notice that the transfer function from  $f(t)$  to  $x(t)$  in  $\ddot{x} + \omega_0^2 x = f(t)$  is  $H(s) = 1/(s^2 + \omega_0^2)$  ... just as in our classroom discussions.
- **JL 1.1 — Integrals as Solutions:** Read this whole section. It's the first section in the whole book, so it should feel like a gentle review. Look through all the problems, but practice seriously on 1.1.6 and 1.1.103.
- **JL 1.2 — Slope Fields:** This is a short and sweet section that should build your intuition. Try 1.2.6 and 1.2.104.
- **JL 1.3 — Separable Equations:** Read through the whole section and solve as many of 1.3.1–1.3.10 and 1.3.101–1.3.105 as time permits.
- **JL 1.4 — Linear equations and the integrating factor:** We met a different approach to solving problem JL(1.3) in class, based on “guessing”  $y(t) = y_1(t)u(t)$  with  $y_1(t) = e^{P(t)}$  for your choice of function  $P(t)$  compatible with  $\dot{P}(t) = p(t)$ . Ultimately the methods are equivalent, and you should use whichever you find easier to work with. But do practice: all the problems in this section are accessible by both techniques.
- **WFT 2.1 — Linear First Order Equations** Start by looking at the plots and diagrams in the section to make sure you know what concepts they show. Then read through problems 1–37. Convince yourself that you know how to approach each one, then actually solve a few of them. Look back to the solved examples in the writeup for helpful hints. Finally, if you get really stuck, actually read the whole section.
- **WFT 2.2 — Separable Equations** In all WFT usage this week, apply the strategy outlined for Section 2.1. In Section 2.2, focus on problems 1–30.
- **WFT 4.2 — Cooling and Mixing** The hard part here is transforming a given story into a differential equation. The key is always *rate balance*: the units of  $dy/dt$  are always the units of  $y$  per unit time (e.g., kg/s or  $^\circ\text{C}/\text{min}$ ), and these units must be matched by the units of every other term. Following this principle and using common sense often gets the job done. Look at problems 1–21.
- **WFT 10.1 — Introduction to Systems of Differential Equations** Key skills here are *Rewriting Higher Order Systems as First Order Systems* (page 512) and especially *Rewriting Scalar Differential Equations as Systems* (page 513). Both are pretty easy, because they do nothing more than change the cosmetic appearance of the problem. Actually *solving* the given differential equation—whether it's a system or not—takes serious effort, but that's a step beyond the transformations shown here. Try problems 1, 2, and 5. (Skip the section on Numerical Methods.)
- **WFT 10.2 — Linear Systems of Differential Equations** This is a gentle intro to the vector-matrix ODE's we are now studying. In this section, Trench doesn't show how to solve anything, he just outlines what to expect and invites readers to check whether a given vector-valued function satisfies a given ODE. If you know how problems 4–5 work, and you can do 2 of the 6 sub-problems they list, you're all set. (Problem 8 is out-of-scope this week, but you might find it interesting nonetheless.)
- **WFT 10.3 — Basic Theory of Homogeneous Linear Systems** This is all fascinating and fundamental. Skip it.
- **WFT 10.4 — Constant Coefficient Homogeneous Linear Systems I** Read the statement (not the proof) of Theorem 10.4.1, and look at the solved Examples 10.4.1–10.4.3. Then stop. If you can do the algebraic work requested in classic  $2 \times 2$  or  $3 \times 3$  examples like problems 1–15, that's enough for this week. The case-by-case analysis and lovely sketches that follow will be our theme for a future week.

**Next Week's Test.** On Thursday 30 October 2014, there will be a 110-minute test starting at 08:00. Out of the 75 marks available in total, the number designated for “Math” is 30. Examinable topics include everything on this sheet, and from previous weeks.