The Rocket Car

UBC Math 403 Lecture Notes by Philip D. Loewen

We consider this simple system with control constraints:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \qquad u \in [-1, 1],$$

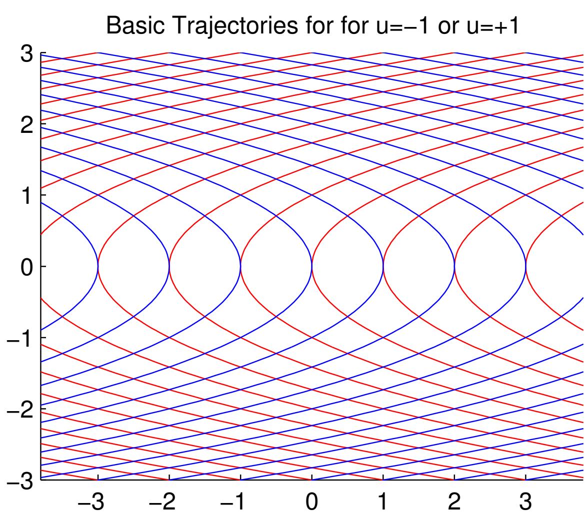
Dynamics. Consider the system evolution on a time interval $[a, b]$ with initial point $x(a) = (\xi_1, \xi_2)$ and constant control $u = \sigma$, where $s = \pm 1$. Under this choice, $\sigma^2 = 1$ and $\sigma = 1/\sigma$. Using the initial conditions, we find

$$\begin{aligned} \dot{x}_2 &= \sigma \qquad \Longrightarrow \qquad x_2 = \xi_2 + \sigma(t-a); \\ \dot{x}_1 &= x_2 \qquad \Longrightarrow \qquad x_1 = \xi_1 + \xi_2(t-a) + \tfrac{1}{2}\sigma(t-a)^2. \end{aligned}$$

To see the shape of these trajectories in the (x_1, x_2)-plane, we solve the first equation for $(t-a) = \sigma(x_2(t) - \xi_2)$ and substitute into the second: it follows that for each t, $(x_1, x_2) = (x_1(t), x_2(t))$ obeys

$$x_1 - \xi_1 = \sigma\xi_2(x_2 - \xi_2) + \tfrac{1}{2}\sigma^3(x_2 - \xi_2)^2 = \frac{\sigma}{2}\left(x_2^2 - \xi_2^2\right).$$

This is a parabola in the phase plane, passing through the given point (ξ_1, ξ_2). Different choices for the initial point give different parabolas, but in all cases the parabolas are congruent curves opening horizontally, with their vertices on the x_1-axis. When $u \equiv \sigma = +1$, the parabolas open to the right and the system state travels upward (since $\dot{x}_2 = \sigma > 0$): some curves like this are shown in red on the figure below. When $u \equiv \sigma = -1$, the parabolas open to the left and the state travels downward: some representatives of this family are shown in blue.



Extremals. Given some $T > 0$, a control function $\widehat{u}\colon [0, T] \to [-1, 1]$ is extremal if it satisfies four conditions involving the pre-Hamiltonian function

$$H(\mathbf{x}, \mathbf{p}, u) = \mathbf{p} \bullet (A\mathbf{x} + Bu) = p_1 x_2 + p_2 u.$$

We list these and expand each one a little.

(a) [The costate equation.] $-\dot{\mathbf{p}}(t) = H_{\mathbf{x}}(\mathbf{x}(t), \mathbf{p}(t), \widehat{u}(t))$ a.e., i.e.,

$$\begin{aligned} -\dot{p}_1(t) &= H_{x_1}(\mathbf{x}(t), \mathbf{p}(t), \widehat{u}(t)) = 0, \\ -\dot{p}_2(t) &= H_{x_2}(\mathbf{x}(t), \mathbf{p}(t), \widehat{u}(t)) = p_1(t). \end{aligned}$$

The first equation implies that $p_1(t)$ must be constant. Then the second equation says $\dot{p}_2(t) = p_1$, so we find that there must be constants m, b such that

$$p_1(t) = -m, \qquad p_2(t) = mt + b.$$

(b) [The state equation.] $\dot{\mathbf{x}}(t) = H_{\mathbf{p}}(\mathbf{x}(t), \mathbf{p}(t), \widehat{u}(t))$ a.e., i.e.,

$$\begin{aligned} \dot{x}_1(t) &= H_{p_1}(\mathbf{x}(t), \mathbf{p}(t), \widehat{u}(t)) = x_2(t), \\ \dot{x}_2(t) &= H_{p_2}(\mathbf{x}(t), \mathbf{p}(t), \widehat{u}(t)) = \widehat{u}(t). \end{aligned}$$

This just confirms our focus on the dynamical system governing the controlled state evolution.

(c) [The maximum condition.] For all but finitely many $t \in [0, T]$,

$$\begin{aligned} H(\mathbf{x}(t), \mathbf{p}(t), \widehat{u}(t)) &\geq H(\mathbf{x}(t), \mathbf{p}(t), u) && \forall u \in [-1, 1], \\ \text{i.e.,} \qquad p_2(t)\widehat{u}(t) &\geq p_2(t)u && \forall u \in [-1, 1], \\ \text{i.e.,} \qquad p_2(t)\left[\widehat{u}(t) - u\right] &\geq 0 && \forall u \in [-1, 1]. \end{aligned}$$

Clearly if $p_2(t) > 0$ this inequality forces $\widehat{u}(t) = +1$, while if $p_2(t) < 0$ it forces $\widehat{u}(t) = -1$. In summary, for all but finitely many times t in $[0, T]$,

$$\widehat{u}(t) = \operatorname{sgn}(p_2(t)) = \operatorname{sgn}(mt + b), \quad \text{where} \quad \operatorname{sgn}(q) \stackrel{\text{def}}{=} \begin{cases} +1, & \text{if } q > 0, \\ -1, & \text{if } q < 0, \\ \text{undefined}, & \text{if } q = 0. \end{cases}$$

This reveals that our study of the system dynamics under the extreme controls was useful work.

(d) [Nontriviality.] $\mathbf{p}(T) \neq \mathbf{0}$. This expands to $(m, mT + b) \neq (0, 0)$. Clearly,

$$(m, mT + b) = (0, 0) \iff (m, b) = (0, 0):$$

negating both statements reveals that $\mathbf{p}(T) \neq \mathbf{0}$ if and only if $(m, b) \neq (0, 0)$.

Attainable Sets. Let's calculate $\mathcal{A}(T; \mathbf{0}, [-1, 1])$, the set of points in (x_1, x_2)-space reachable by all controlled trajectories that start from the origin and travel for time T. We will use $\mathcal{A}$ as shorthand for this set. Recall that any control $\widehat{u}(\cdot)$ whose corresponding state $\mathbf{x}(\cdot)$ terminates on the boundary of $\mathcal{A}$ must be extremal. To build a catalogue of extremal controls, we must synthesize properties (a)–(d) above. Clearly the "switching function" $p_2(t) = mt + b$ is the key to everything: depending on the values of m and b, p_2 might start positive and change signs, start negative and change signs, or keep the same sign for all t in $(0, T)$. Each possibility must be considered.

Case 1: $b \geq 0$, $mT + b \geq 0$. Here $p_2(t) > 0$ for all $t \in (0, T)$ so $\widehat{u}(t) = 1$ for all $t \in (0, T)$, and the system follows the trajectory

$$x_1(t) = \tfrac{1}{2}t^2, \quad x_2(t) = t, \qquad t \in [0, T].$$

The final point in phase space is $(\frac{1}{2}T^2, T)$.

Case 2: $b \leq 0$, $mT + b \leq 0$. Here $p_2(t) < 0$ for all $t \in (0, T)$ so $\widehat{u}(t) = -1$ for all $t \in (0, T)$, and the system follows the trajectory

$$x_1(t) = -\tfrac{1}{2}t^2, \quad x_2(t) = -t, \qquad t \in [0, T].$$

The final point in phase space is $(-\frac{1}{2}T^2, -T)$.

Case 3: $b > 0$, $mT + b < 0$; Case 4: $b < 0$, $mT + b > 0$. Here p_2 changes sign at the instant $a = -b/m$, and we can use the $\sigma = \pm 1$ trick to do all the algebra just once. We just work under the assumption that

$$\sigma b > 0, \quad \sigma(mT + b) < 0:$$

then $\sigma = +1$ corresponds to Case 3, and $\sigma = -1$ is Case 4. The system path has a two-part structure: on an initial segment, $\widehat{u}(t) = \sigma$, giving

$$x_1(t) = \tfrac{1}{2}\sigma t^2, \quad x_2(t) = \sigma t, \qquad t \in [0, a].$$

At the "switching time" a, the system is at $\mathbf{x}(a) = \sigma\left(\frac{1}{2}a^2, a\right)$, and the control switches to the constant $-\sigma$ for $a < t < T$: we deduce that for $a \leq t \leq T$,

$$\begin{aligned} x_1(t) &= \xi_1 + \xi_2(t - a) - \tfrac{1}{2}\sigma(t - a)^2 && [\xi_1 = \tfrac{1}{2}\sigma a^2] \\ x_2(t) &= \xi_2 - \sigma(t - a) = \sigma a - \sigma(t - a) = \sigma(2a - t) && [\xi_2 = \sigma a]. \end{aligned}$$

When we plug in $t = T$ here, these equations reveal the system's final state in terms of two parameters: the initial control value σ, and the switching time a. We can reduce to a single parameter by eliminating a: the second equation gives

$$a = \tfrac{1}{2}(T + \sigma x_2(T)), \quad \text{so} \quad T - a = \tfrac{1}{2}(T - \sigma x_2(T)).$$

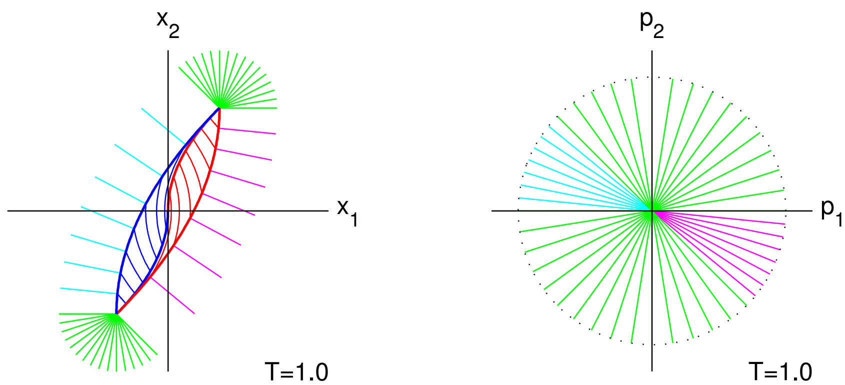
Using this in the first equation, with (x_1, x_2) as shorthand for the more explicit form $(x_1(T), x_2(T))$, we have

$$\begin{aligned} x_1 &= \tfrac{1}{2}\sigma a^2 + \sigma a(T - a) - \tfrac{1}{2}\sigma(T - a)^2 \\ &= \frac{\sigma}{2}\left[a^2 + 2a(T - a) + (T - a)^2 - 2(T - a)^2\right] \\ &= \frac{\sigma}{2}\left[(a + (T - a))^2 - 2\left(\frac{T - \sigma x_2}{2}\right)^2\right]. \\ &= \frac{\sigma}{2}\left[T^2 - \tfrac{1}{2}(T - \sigma x_2)^2\right]. \end{aligned}$$

For fixed $T > 0$ and $\sigma = \pm 1$, this is another parabola in the phase plane.

$$\sigma = 1 \implies x_1 = \frac{1}{2}\left[T^2 - \frac{1}{2}(x_2 - T)^2\right], \quad \text{vertex } \left(\tfrac{1}{2}T^2, T\right);$$
$$\sigma = -1 \implies x_1 = -\frac{1}{2}\left[T^2 - \frac{1}{2}(x_2 + T)^2\right], \quad \text{vertex } \left(-\tfrac{1}{2}T^2, -T\right).$$

The figure below shows the two parabolas just found. The one for $\sigma = +1$ is red, and forms the right edge of the lens-shaped region in (x_1, x_2)-space; the one for $\sigma = -1$ is blue, and forms the left edge. These two parabolas form the boundary of the convex set $\mathcal{A}(T, \mathbf{0}, U)$. (The sketch also shows some outward normals to $\mathcal{A}$ in (x_1, x_2)-space. We will discuss these below.)



Geometry. In the proof that every boundary trajectory was generated by an extremal control, we showed that the function $\mathbf{p}$ in the extremality condition obeys

$$\mathbf{p}(T) \perp \mathcal{A}(T; \mathbf{0}, U) \text{ at } x(T).$$

We also noted that there is some redundancy in the extremality definition: a function $\mathbf{p}$ will make its four conditions work if and only if the function $\lambda\mathbf{p}$ will do likewise for every constant $\lambda > 0$. Thus it is legitimate and convenient to assume that $\mathbf{p}(T)$ is a unit vector, and concentrate on its slope,

$$\frac{p_2(T)}{p_1(T)} = \frac{mT + b}{(-m)} = -\frac{b}{m} - T = a - T, \quad \text{where} \quad a = -\frac{b}{m}. \qquad (**)$$

The sketch above shows various possible unit vectors $\mathbf{p}(T)$ in (p_1, p_2)-space, and the colour codes suggest a correspondence between these vectors and the corresponding outward normals to $\mathcal{A}(T)$ pictured in (x_1, x_2)-space. To make this correspondence quite precise, let us review the four cases considered above.

Case 1: $b \geq 0$, $mT + b \geq 0$. In this case, $p_2(T) = mT + b \geq 0$.

Subcase 1A: We certainly land in Case 1 if both $m \geq 0$ and $b \geq 0$. When $m > 0$, this gives $a = -b/m \leq 0$, so the slope shown in $(**)$ above is negative and not larger than $-T$. Unit vectors $\mathbf{p}(T)$ with $p_2(T) > 0$ and slopes in this range contribute a small wedge of vectors in the second quadrant of (p_1, p_2)-space. These are coloured green in the sketch above.

Subcase 1B: The other possibility in Case 1 is that $m < 0$ and $b \geq 0$, but $b+mT \geq 0$. Rearranging this inequality (remember $m < 0$) gives $T \leq -b/m = a$, i.e., $a - T \geq 0$. Vectors with this slope fill the whole first quadrant in (p_1, p_2)-space. These are also coloured green in the sketch above.

All the green vectors in the half-plane $p_2 \geq 0$ provide outward normals to the set $\mathcal{A}$ at the point $(\frac{1}{2}T^2, T)$.

Case 2: $b \leq 0$, $mT + b \leq 0$. In this case, $p_2(T) = mT + b \leq 0$, and the analysis is quite symmetric. When both $m \leq 0$ and $b \leq 0$ (subcase 2A), we find vectors of slope more negative than $-T$ but having $p_2(T) < 0$: these produce a wedge in the fourth quadrant of (p_1, p_2)-space that we have coloured green in the sketch. The only other possibility is that $m > 0$ and $b \leq 0$, but still $p_2(T) \leq 0$. This happens whenever vector $\mathbf{p}(T)$ points into the fourth quadrant of (p_1, p_2)-space. Combining this wedge with the previous one gives a cone of vectors in the half-space $p_2 \leq 0$ that is precisely $N_{\mathcal{A}}(-\frac{1}{2}T^2, -T)$.

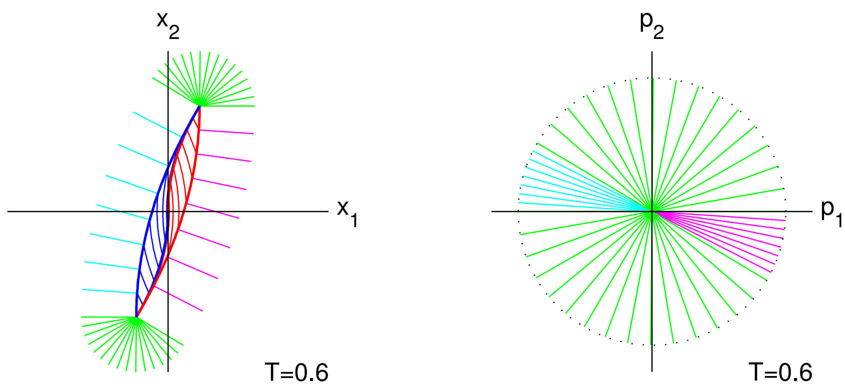
Case 3: $b > 0$, $mT + b < 0$. This requires $m < 0$, so $p_1(T) = -m > 0$ and $a = -b/m > 0$. Also,

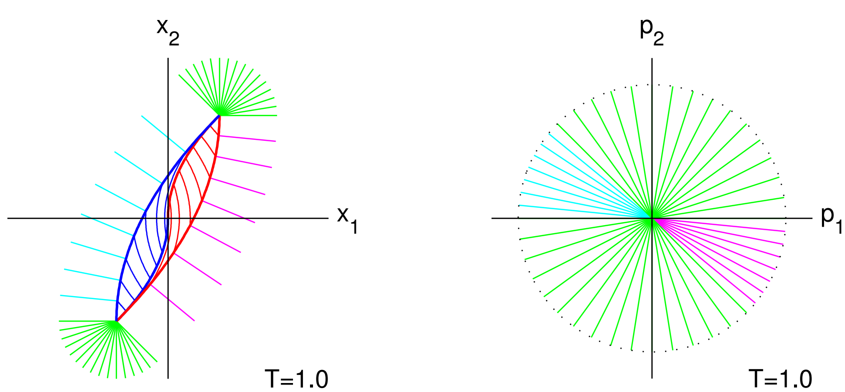
$$mT + b < 0 \iff T - (-b/m) > 0 \iff T > a.$$

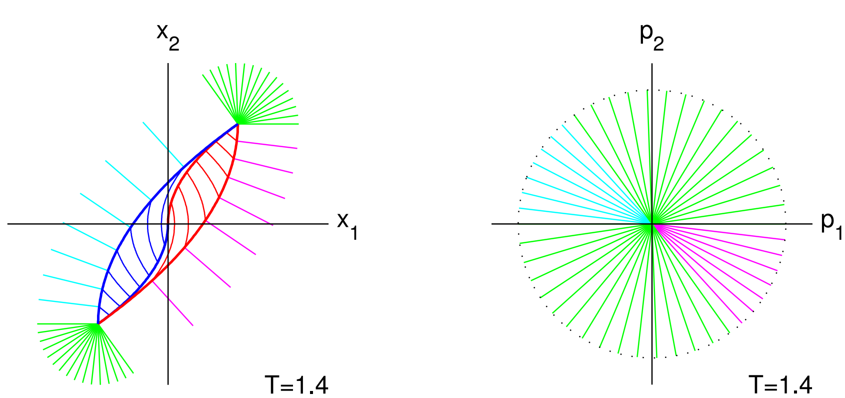
This is no surprise: we recognize $a = -b/m$ in $(**)$ as the switching time determined by function $\mathbf{p}$ in our previous exploration of Case 3. But now the inequality $0 < a < T$ implies that the slope (namely, $a-T$) in $(**)$ occupies the range $-T < a-T < 0$. This identifies a cone of vectors in the fourth quadrant of (p_1, p_2)-space that is coloured magenta in the sketch. Each direction in this cone identifies a unique point $\mathbf{x}(T)$ on the boundary of $\mathcal{A}$, and this is precisely the point generated by using control $u \equiv 1$ in $[0, a)$ and then switching to $u \equiv -1$ on $(a, T]$.

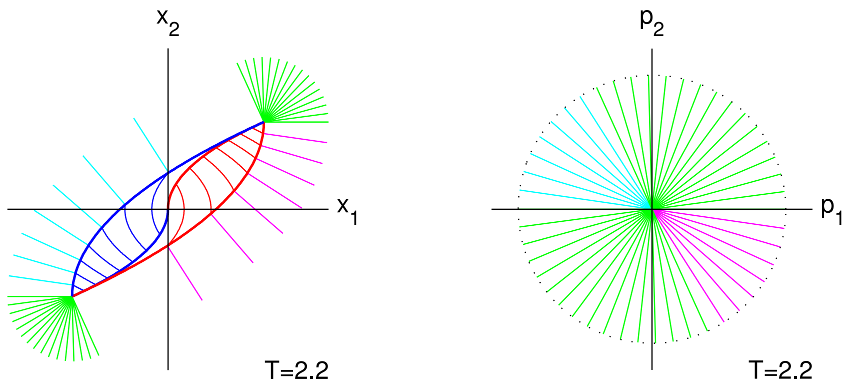
Case 4: $b < 0$, $mT + b > 0$. This requires $m > 0$, so $p_1(T) = -m < 0$ and $a = -b/m > 0$. By analysis similar to that for Case 3, we find that the slope $a - T$ of vectors arising here also lies in the interval $(-T, 0)$, only this time $p_2(T) > 0$ so we pick up only the corresponding cone of vectors in the second quadrant. This is shown in cyan on the sketch above. Every cyan vector identifies a unique point $\mathbf{x}(T)$ on the (blue) boundary of $\mathcal{A}$, namely, the point generated by using control $u \equiv -1$ in $[0, a)$ and then switching to $u \equiv 1$ on $(a, T]$.

Evolution of the Attainable Set. Everything said so far applies to the construction of $\mathcal{A}(T)$ for a fixed final time $T > 0$. Now imagine how $\mathcal{A}(T)$ changes as T increases. The coloured sectors of (p_1, p_2)-space are separated by the p_1-axis and the diagonal line $p_2 = -Tp_1$: as T increases, this line gets steeper and steeper. The green cones get squeezed closer and closer to just the first and third quadrants, while the cyan and magenta cones widen. This is reflected in the shape of the attainable sets, which grow both vertically and horizontally. However, their horizontal growth is more rapid than their vertical growth for large T. The figures below shows snapshots of the general construction outlined above for various final times.

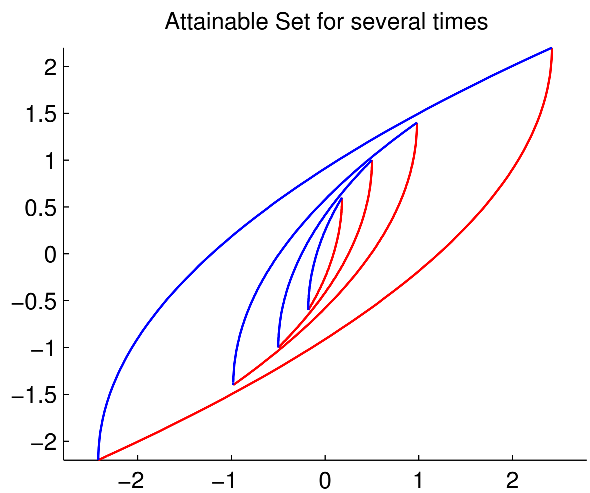








The axis scaling in (x_1, x_2)-space is different in each of the snapshots above, to show most clearly how the shape and orientation of the set $\mathcal{A}(T)$ evolves. But of course the set $\mathcal{A}(T)$ is also growing. The sketch below shows the attainable sets above on the same set of axes.



The Minimum-Time Problem. For a given initial vector ξ, consider the problem of steering the system point (x_1, x_2) from $\xi = (\xi_1, \xi_2)$ to the origin as rapidly as possible. In simple terms, we seek the smallest $T > 0$ for which some admissible trajectory $\mathbf{x}$ with $\mathbf{x}(0) = \xi$ obeys $\mathbf{x}(T) = 0$. In terms of the attainable set, we are looking for the smallest T satisfying

$$\mathbf{0} \in \mathcal{A}(T; \xi, U). \qquad (*)$$

A detailed analysis of the way the sets $\mathcal{A}(t; \xi, U)$ evolve as t increases will show that the following scenario is impossible:

$$\forall t \in [0, T),\ \mathbf{0} \notin \mathcal{A}(t), \quad \text{and} \quad \mathbf{0} \in \operatorname{int} \mathcal{A}(T). \qquad (**)$$

In other words, an optimizing state trajectory $\mathbf{x}(\cdot)$ with minimum time T must be one for which

$$\mathbf{x}(T) \in \operatorname{bdy} \mathcal{A}(T; \xi, U).$$

As we have seen, this happens if and only if the trajectory $\mathbf{x}(\cdot)$ is generated by an *extremal* control $\widehat{u}(\cdot)$.

Working Backward.* We know that every extremal control takes the form

$$\widehat{u}(t) = \begin{cases} -\sigma, & \text{if } 0 \le t < a, \\ +\sigma, & \text{if } a < t \le T, \end{cases}$$

for some $a \in [0, T]$, where σ is either -1 or $+1$. To analyze the details, work backwards from the target: since $\widehat{u} \equiv \sigma$ on the final interval, and the system trajectory ends at $(0, 0)$, the final segment of the system's path in phase space must lie on the parabolic segment

$$S_\sigma: \qquad x_1 = \frac{\sigma}{2} x_2^2, \quad \sigma x_2 \le 0, \tag{1}$$

for all t in $[a, T]$. The detailed time-dependence for this segment is given by

$$x_1(t) = \frac{\sigma}{2}(t - T)^2, \quad x_2(t) = \sigma(t - T), \qquad a \le t \le T.$$

For initial points ξ lying right on the curve S_σ, the constant control $\widehat{u} \equiv \sigma$ provides an extremal transfer to the origin with no switch at all ($a = 0$). In the vast majority of cases, however, a switch is required: the system trajectory reaches the curve S_σ by travelling along one of the parabolas compatible with $u \equiv -\sigma$, i.e.,

$$x_1 = -\frac{\sigma}{2} x_2^2 + k. \tag{2}$$

Motion along such a parabola will reach S_σ if and only if $\sigma k > 0$.

Feedback Synthesis. Thinking about our findings above lets us determine the control value that will be used at every point of phase space. Define

$$s(z) = -\sqrt{2|z|}\,\mathrm{sgn}(z), \qquad z \in \mathbb{R},$$

so that the curve $S = S_{-1} \cup S_{+1}$ in phase space can be represented simply as $x_2 = s(x_1)$. Then any extremal control $\widehat{u}$ on any interval, for any initial point, will satisfy the identity

$$\widehat{u}(t) = \widehat{U}(\mathbf{x}(t)), \qquad \text{where} \qquad \widehat{U}(x_1, x_2) = \begin{cases} -1, & \text{if } x_2 > s(x_1) \text{ or } (x_1, x_2) \in S_{-1}, \\ +1, & \text{if } x_2 < s(x_1) \text{ or } (x_1, x_2) \in S_{+1}, \\ 0, & \text{if } (x_1, x_2) = (0, 0). \end{cases}$$

Most of these ingredients can be found in the sketch below. The S-shaped curve dividing the plane is made of the two parts S_{-1} and S_{+1}. Trajectories drawn in blue are generated by the control value -1: the system point moves downward along

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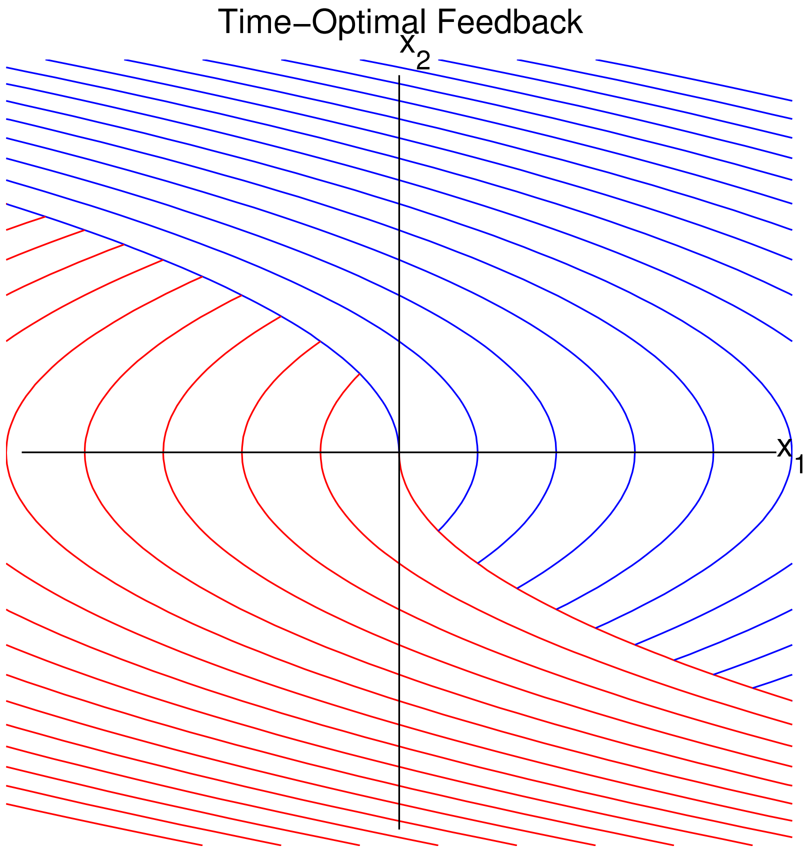
* It's probably easiest to read this section assuming $\sigma = +1$ throughout on the first time through.

them. Trajectories coloured red are produced by the control +1: the system moves upward along them. The feedback function $\widehat{U}$ expresses these choices algebraically.



Details. It's instructive to practice finding all the details of the time-optimal trip to the origin for a given starting point (ξ_1, ξ_2). Assuming $\xi_2 > s(\xi_1)$, the initial part of the trajectory will follow a parabola as in (2), with $\sigma = 1$. The constant k must match the initial state, i.e., $k = \xi_1 + \frac{1}{2}\xi_2^2$, and the initial segment is

$$x_1 = \xi_1 + \frac{1}{2}\left[\xi_2^2 - x_2^2\right]. \tag{1}$$

This intersects S_{+1} at a point (x_1, x_2) that satisfies

$$\xi_1 + \frac{1}{2}\left[\xi_2^2 - x_2^2\right] = \frac{1}{2}x_2^2, \quad \text{i.e.,} \quad x_2^2 = \xi_1 + \tfrac{1}{2}\xi_2^2.$$

Since $x_2 < 0$ at all points of S_{+1}, the switch occurs at the point where

$$x_2 = -\sqrt{\xi_1 + \tfrac{1}{2}\xi_2^2}, \qquad x_1 = \tfrac{1}{2}x_2^2 = \frac{1}{2}\xi_1 + \frac{1}{4}\xi_2^2.$$

Since elapsed time equals vertical displacement in this model, the time from start to switch is

$$a = \xi_2 + \sqrt{\xi_1 + \tfrac{1}{2}\xi_2^2},$$

and the time from the switch to the target is

$$T - a = \sqrt{\xi_1 + \tfrac{1}{2}\xi_2^2}.$$

Hence the total time for this extremal transfer is

$$T = a + (T - a) = \xi_2 + 2\sqrt{\xi_1 + \tfrac{1}{2}\xi_2^2}.$$

Minimum Time to a Target Set. Time is short. For now, we just give examples.

Example. Consider the rocket car, for which the dynamics are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u, \qquad u \in [-1, 1]. \end{aligned}$$

Use the Maximum Principle to derive a feedback strategy which steers any initial point to the x_1-axis (i.e., stops the car) in minimum time. (The solution is intuitively obvious, of course—we are interested in whether it is also an obvious consequence of our theory.)

Solution. Describe the x_1-axis as the target set $S = \{(x_1, x_2) : x_2 = 0\}$ and recall the geometric interpretation of the Maximum Principle. This asserts that the final costate vector $-p(T)$ must be an outward normal to the target set S at the final point $x(T)$. Now the nonzero vectors normal to S at any point are precisely those whose first component is 0, so the solution to the adjoint equation reduces to

$$p_1(t) \equiv 0, \quad p_2(t) \equiv b$$

for some constant $b \neq 0$. Consequently every extremal control $\widehat{u}(t) = \mathrm{sgn}(p_2(t))$ must be constant with value either -1 or $+1$. To meet the boundary conditions, we must choose $\widehat{u}(t) = +1$ if $x_2(t) < 0$, and $\widehat{u}(t) = -1$ if $x_2(t) > 0$. That is, the time-optimal feedback strategy is simply to use maximum braking acceleration until the car stops. This accords well with the intuition.

Example. Consider the rocket car discussed in class, with the dynamics

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u, \qquad u \in [-1, 1]. \end{aligned}$$

Use the Maximum Principle to derive a feedback strategy which steers any initial point to the closed unit ball $\mathbb{B}[\mathbf{0}; 1]$ of $\mathbb{R}^2$ in minimum time. Use parametric equations to describe the switching curves.

Solution. The key to this problem is the transversality condition in PMP. The target set here is $S = \mathbb{B}[0; 1]$—a closed convex set, whose boundary is parametrized by $(\cos\theta, \sin\theta)$, $\theta \in (-\pi, \pi]$. Let us suppose that some optimal control $\widehat{u}$ defined on some interval $[0, T]$ directs the system's evolution so that its first contact with the target set is at the point $(x_1(T), x_2(T)) = (\cos\theta, \sin\theta)$. Then $\widehat{u}$ and the corresponding state x must satisfy the usual conditions of the maximum principle, together with the additional condition that the final value of the costate vector be the negative of an outward normal direction to the target set at the optimal endpoint. Now the unit vector perpendicular to $\mathbb{B}[0;1]$ at $(\cos\theta, \sin\theta)$ is precisely $(\cos\theta, \sin\theta)$: thus we have the transversality condition

(d) $[-p_1(T) \quad -p_2(T)] = [\cos\theta \quad \sin\theta]$.

Our solution is based on a case-by-case analysis of the optimal trajectories, parametrized by the angle θ identifying the point where they first make contact with the target.

Recall that $p_1 = -\cos\theta$ is constant, while $p_2(t) = p_2(T) - (t-T)p_1 = -\sin\theta + (t-T)\cos\theta$. Consequently the time τ at which p_2 changes sign must be

$$\tau = T + \tan\theta, \quad \text{provided } \theta \neq \pi/2,\ 3\pi/2.$$

For $\theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$, this gives $\tau \geq T$, so the control $\widehat{u}$ must be constant. When $\theta = \pi/2$ or $\theta = 3\pi/2$, then $p_2(t) = -\sin\theta$ is a constant with value ± 1: hence $\widehat{u}$ is constant in these cases also.

Case $0 \leq \theta \leq \pi/2$. Here $p_2(t) < 0$ for all $t \in [0, T)$, so $\widehat{u}(t) \equiv -1$ and the system reaches $\mathbb{B}[0;1]$ along a parabolic arc of the form

$$x_1 = -\tfrac{1}{2}x_2^2 + \ell, \quad x_2 \geq 0.$$

The trajectories corresponding to $\theta = 0$ and $\theta = \pi/2$ are given by

$$x_1 = \tfrac{1}{2} - \tfrac{1}{2}x_2^2, \quad x_2 \geq 1 \qquad (\text{for } \theta = 0),$$
$$x_1 = 1 - \tfrac{1}{2}x_2^2, \quad x_2 \geq 0 \qquad (\text{for } \theta = \pi/2);$$

all points between these curves reach $\mathbb{B}[0;1]$ in minimum time along similar parabolas, using the constant control $\widehat{u} = -1$.

Case $\pi \leq \theta \leq 3\pi/2$. Here $p_2(t) > 0$ for all $t \in [0, T)$, so $\widehat{u}(t) \equiv +1$ and the system reaches $\mathbb{B}[0;1]$ along a parabolic arc of the form

$$x_1 = \tfrac{1}{2}x_2^2 + d, \quad x_2 \leq 0.$$

The trajectories corresponding to $\theta = \pi$ and $\theta = 3\pi/2$ are given by

$$x_1 = -1 + \tfrac{1}{2}x_2^2, \quad x_2 \leq 0 \qquad (\text{for } \theta = \pi),$$
$$x_1 = -\tfrac{1}{2} + \tfrac{1}{2}x_2^2, \quad x_2 \leq -1 \qquad (\text{for } \theta = 3\pi/2);$$

all points between these curves reach $\mathbb{B}[0;1]$ in minimum time along similar parabolas, using the constant control $\widehat{u} = +1$.

Case $\pi/2 < \theta \leq \pi$. Now $\tau = T + \tan\theta < T$, and $p_2(T) = -\sin\theta < 0$. Hence $\widehat{u}(t) \equiv -1$ on $(\tau, T]$, and it follows that for $t \in [\tau, T]$,

$$x_2(t) = T - t + \sin\theta,$$
$$x_1(t) = -\tfrac{1}{2}(T-t)^2 + (t-T)\sin\theta + \cos\theta.$$

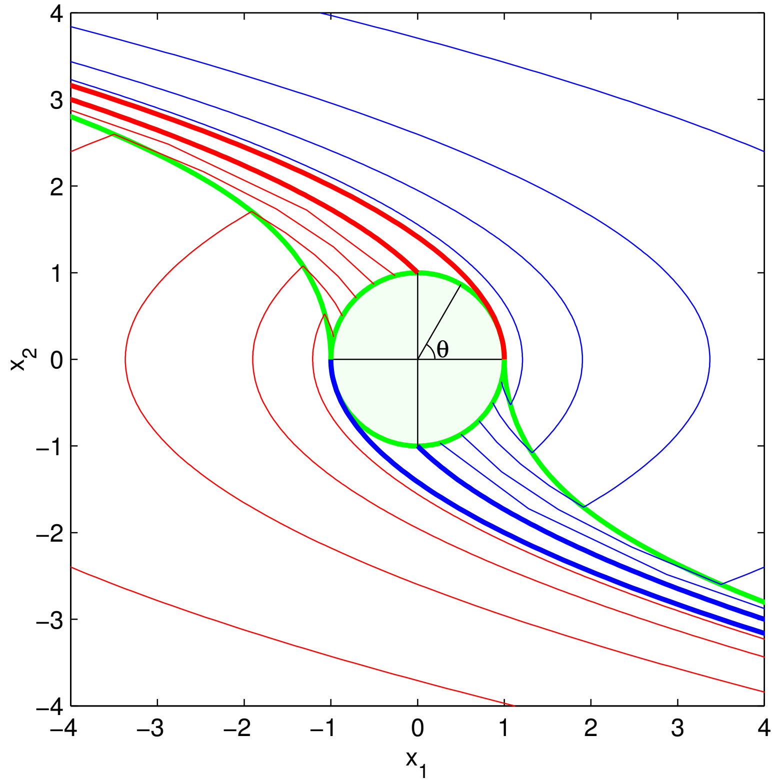


Figure 1: Time-optimal trajectories and switching curves

The position of the system when the switch occurs at time $t = \tau = T + \tan\theta$ is

$$x_1(\tau) = -\tfrac{1}{2}\tan^2\theta + \sin\theta\tan\theta + \cos\theta$$
$$x_2(\tau) = \sin\theta - \tan\theta.$$

This pair of parametric equations involving $\theta \in (\pi/2, \pi)$ describes a curve we call S_- in the (x_1, x_2)-plane. Notice that the parametric equations imply

$$x_1 = -\tfrac{1}{2}x_2^2 + \tfrac{1}{2}\sin^2\theta + \cos\theta,$$

so the curve S_- approximates the parabola $x_1 = -\frac{1}{2}x_2^2 + \frac{1}{2}$ for values of θ near $\pi/2$: this parabola describes the boundary of the set of initial values covered by the case where $0 \le \theta \le \pi/2$.

Points lying on or above the curve S_- have $\tau \le 0$: they travel to $\mathbb{B}[0;1]$ along parabolic arcs using the constant input $\widehat{u} \equiv -1$. Points lying below S_- have $0 < \tau < T$: they travel to S_- along parabolic arcs using the constant input $\widehat{u} \equiv 1$, then switch to $\widehat{u} \equiv -1$ and follow the resulting parabolic path to $\mathbb{B}[0;1]$.

Case $3\pi/2 < \theta \le 2\pi$. Now $\tau = T + \tan\theta < T$, and $p_2(T) = -\sin\theta > 0$. Hence $\widehat{u}(t) \equiv +1$ on $(\tau, T]$, and it follows as in the previous case that

$$x_1(\tau) = \tfrac{1}{2}\tan^2\theta + \sin\theta\tan\theta + \cos\theta$$
$$x_2(\tau) = \sin\theta + \tan\theta.$$

This pair of parametric equations involving $\theta \in (3\pi/2, 2\pi)$ describes a curve we call S_+ in the (x_1, x_2)-plane. Notice that the parametric equations imply

$$x_1 = \tfrac{1}{2}x_2^2 - \tfrac{1}{2}\sin^2\theta + \cos\theta,$$

so the curve S_+ approximates the parabola $x_1 = \frac{1}{2}x_2^2 - \frac{1}{2}$ for values of θ near $3\pi/2$: this parabola describes the boundary of the set of initial values covered by the case where $\pi \le \theta \le 3\pi/2$.

Points lying on or below the curve S_+ have $\tau \le 0$: they travel to $\mathbb{B}[0;1]$ along parabolic arcs using the constant input $\widehat{u} \equiv +1$. Points lying above S_+ have $0 < \tau < T$: they travel to S_+ along parabolic arcs using the constant input $\widehat{u} \equiv -1$, then switch to $\widehat{u} \equiv +1$ and follow the resulting parabolic path to $\mathbb{B}[0;1]$.

Summary. The optimal feedback law may be described in terms of the set $S = S_- \cup \mathbb{B}[0;1] \cup S_+$:

$$\widetilde{u}(x) = \begin{cases} -1, & \text{if } x \text{ lies above } S \text{ or on } S_-, \\ +1, & \text{if } x \text{ lies below } S \text{ or on } S_+. \end{cases}$$

A detailed sketch of all four cases is provided.

Discussion. Consider the minimum-time problem with the specific initial point $(x_1, x_2) = (-4, 0)$. The feedback law found above applies control $u = +1$ to drive the system along the parabola $x_1 = \frac{1}{2}x_2^2 - 4$ until it hits S_-. To find the intersection point, one substitutes the parametric forms from line $(*)$ and solves numerically for $\theta \approx 2.2864$. Hence the intersection point is approximately at $(-2.1857, 1.9049)$, and from this point the control $u = -1$ drives the system along another parabola to hit the unit circle at $(\cos\theta, \sin\theta) \approx (-0.6561, 0.7547)$. The total time elapsed is the sum of the vertical displacements along the two parabolas in question:

$$T_{\min}(-4, 0) \approx [1.9049 - 0.0000] + [1.9049 - 0.7547] = 3.0551.$$

This contrasts favourably with a popular suboptimal strategy of switching along the curve $x_1 = -1 - \frac{1}{2}x_2^2$. This provides the minimum-time path from the initial point $(-4, 0)$ to the one-point target at $(-1, 0)$. Calculation shows that this strategy switches at the point $(x_1, x_2) = (-5/2, \sqrt{3})$, and reaches its destination in time

$$T_{\mathrm{hit}(-1,0)}(-4, 0) = [\sqrt{3} - 0] + [\sqrt{3} - 0] \approx 3.4641.$$