Write your answers in the booklet provided. Start each solution on a separate page.

OPEN BOOK EXAM. SHOW ALL YOUR WORK!!

[5] **1.** Suppose the time-varying vectors $\mathbf{x} = \mathbf{x}(t)$ and $\mathbf{p} = \mathbf{p}(t)$ satisfy

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t), \qquad \dot{\mathbf{p}}(t) = -A(t)^T\mathbf{p}(t), \qquad \text{a.e. } t.$$

Prove: If one has $\mathbf{x}(t_0) \perp \mathbf{p}(t_0)$ at some instant t_0, then in fact $\mathbf{x}(t) \perp \mathbf{p}(t)$ for all t.

[15] **2.** (a) Find the matrix e^{tA}, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(b) Let $\mathbf{x}(t;\xi)$ denote the unique solution of $\dot{\mathbf{x}} = A\mathbf{x}$ satisfying $\mathbf{x}(0) = \xi$. Prove:

(i) There are initial states ξ in $\mathbb{R}^3$ for which $|\mathbf{x}(t;\xi)| \to \infty$ as $t \to \infty$.

(ii) There are initial states ξ in $\mathbb{R}^3$ for which $|\mathbf{x}(t;\xi)| \to 0$ as $t \to \infty$.

(iii) Every ξ in $\mathbb{R}^3$ is described by either (i) or (ii).

Construct a simple mathematical test that uses the components of $\xi = (\xi_1, \xi_2, \xi_3)$ to determine whether case (i) or case (ii) applies.

[15] **3.** Consider this single-input system with a three-dimensional state:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t), \qquad \text{a.e. } t. \qquad (*)$$

(a) Is this system controllable? Justify your answer.

(b) Find the attainable set $\mathcal{A}(t;0)$ for $t > 0$. (There are no control constraints.)

(c) A client asks you to supply a "feedback" matrix $F = [\,a \quad b \quad c\,]$ such that substituting $u = F\mathbf{x}$ in $(*)$ will produce an autonomous system for which every trajectory $\mathbf{x}$ obeys $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$. Choose one of the following responses, and support it with detailed calculations or a clear proof, as appropriate:

(i) "Here is a matrix of the type you requested. Thank you for your business."

(ii) "Sorry, a matrix meeting your specifications does not exist. We cannot accept your order."

[15] **4.** Suppose the scalar-valued function $\widehat{u}$ is an extremal control for the single-input system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad -1 \le u(t) \le 1, \qquad \text{a.e. } t \in [0,1].$$

Prove: If there is some open subinterval (a,b) of $[0,1]$ such that $|\widehat{u}(t)| < 1$ for each $t \in (a,b)$), then the matrix pair (A,B) must fail to be controllable.