Write your answers in the booklet provided. Start each solution on a separate page.

OPEN BOOK EXAM. SHOW ALL YOUR WORK!!

1. A certain physical system involves two equal masses whose deviations from equilibrium obey

$$\begin{aligned} \ddot{y}_1 &= -(1+\alpha)y_1 + y_2 + u \\ \ddot{y}_2 &= y_1 - (1+\beta)y_2 \end{aligned} \tag{1}$$

Here α and β are positive real constants, and u is the force applied to the first mass.

(a) Transform the given dynamics into a 4×4 system of first-order linear equations.

(b) Find all parameter pairs (α, β) [if any] for which the system found in (a) is controllable.

(c) Suppose $u \equiv 0$. Would detailed knowledge of $y_2(t)$ for each instant in some time interval of length 1 be enough to uniquely determine the evolution of the full 4-dimensional state for all later times? Justify your answer (which may depend on α and β) with a calculation, not just words.

2. Consider the controlled linear system $\dot{\mathbf{x}} = A\mathbf{x} + Bu$ in which

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1/2 & 0 & -1/2 \\ -2 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}. \tag{2}$$

(a) Write the system of controlled ODE's satisfied by $\mathbf{y} \overset{\text{def}}{=} P^{-1}\mathbf{x}$, given

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}; \qquad \text{note that } P^{-1} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \end{bmatrix}.$$

(b) Find a state feedback matrix F for the original system that puts the eigenvalues of $A + BF$ at -1, $-1-i$, $-1+i$.

3. Consider the following nonlinear dynamical system with control input w:

$$\begin{aligned} \dot{y}_1 &= \sin(y_1 y_2), \\ \dot{y}_2 &= -y_1 \cos(y_2) + \frac{w}{1+w^2}. \end{aligned} \tag{3}$$

(a) Explain why $(\overline{y}_1, \overline{y}_2) = (2, \pi/2)$ is an equilibrium point when $w \equiv \overline{w} \overset{\text{def}}{=} 0$.

(b) Linearize system (3) around the equilibrium point in (a). That is, find a constant-coefficient system of the standard form $\dot{\mathbf{x}} = A\mathbf{x} + Bu$ whose behaviour near $\mathbf{x} = \mathbf{0}$ and $u = 0$ a good approximation for the behaviour of system (3) near $(\overline{y}_1, \overline{y}_2)$ and $\overline{w}$.

(c) Is it possible to choose a constant k for which the feedback law $u = kx_1$ stabilizes the system in (b)? If so, find one and explain why it works; if not, explain why not.