Math 403 Problem Set 9

Due in class on Wednesday 14 April 2010

1. Choose constant a, m, n so that the function $V(x, y) = x^m + ay^n$ is decreasing along every nonconstant trajectory of the planar system

$$\dot{x} = -x + 2y^3 - 2y^4, \qquad \dot{y} = -x - y + xy.$$

Deduce that this system can have no periodic orbits.

2. Consider this nonlinear dynamical system involving a nonzero constant α:

$$\dot{x} = \alpha x^2 - \alpha y, \qquad \dot{y} = 6x - x^2 - xy.$$

(a) Find the unique equilibrium point for this system in the region $x > 0$, $y > 0$. Call it $(\overline{x}, \overline{y})$.

(b) Assess the stability of the equilibrium point $(\overline{x}, \overline{y})$. Express your answer in terms of α.

(c) Let $\alpha = -1$. Find a radius $r > 0$ such that $\mathbb{B}_r(\overline{x}, \overline{y})$ is flow-invariant for the system, and any trajectory starting in $\mathbb{B}_r(\overline{x}, \overline{y})$ converges to $(\overline{x}, \overline{y})$ as $t \to \infty$. Here we have used the notation

$$\mathbb{B}_r(\overline{x}, \overline{y}) = \left\{(x, y) \,:\, (x - \overline{x})^2 + (y - \overline{y})^2 < r^2\right\}$$

for the open disk of radius $r > 0$ and centre $(\overline{x}, \overline{y})$.

3. Consider the following three-dimensional system involving a constant $\varepsilon \geq 0$:

$$\dot{x} = y(1 - z), \quad \dot{y} = -x(1 - z) - \varepsilon y, \quad \dot{z} = -z(1 - z).$$

(i) Prove that the origin is a stable equilibrium point for every $\varepsilon \geq 0$.

(ii) Prove that the origin is asymptotically stable whenever $\varepsilon > 0$.

(iii) Prove that the origin is **not** asymptotically stable when $\varepsilon = 0$. (Clue: Investigate trajectories starting in the (x, y)-plane.)

(iv) Describe the set of initial points $\xi \in \mathbb{R}^3$ for which $\mathbf{x}(t; \xi) \to \mathbf{0}$ as $t \to \infty$.
[Here $\mathbf{x} = (x, y, z)$ and $t \mapsto \mathbf{x}(t; \xi)$ is the system trajectory starting from $\mathbf{x}(0; \xi) = \xi$.]

4. Consider the second-order system

$$\ddot{x} + 2\dot{x} + x(1 - x^2) = 0.$$

Use an energy-like function V to find a region of asymptotic stability for the constant solution $x(t) = 0$. (Here a *region of asymptotic stability* means an open set Ω in $(x, \dot{x})$-space with $(0, 0) \in \Omega$, such that if $(x(0), \dot{x}(0)) \in \Omega$, then $(x(t), \dot{x}(t)) \in \Omega$ for all $t \geq 0$ and $(x(t), \dot{x}(t)) \to (0, 0)$ as $t \to \infty$.)

Continued . . .

5. Consider the classic "double integrator" $\ddot{x} = u$, expressed via

$$(*) \qquad \left[\begin{array}{l} \dot{x}_1 = x_2, \\ \dot{x}_2 = u. \end{array}\right.$$

(a) Show that specifying $u(t) = U(x_1(t), x_2(t))$ in $(*)$, where $U(x_1, x_2) = -(x_1 + x_2)$, produces a system for which the origin is an asymptotically stable equilibrium point which has all of $\mathbb{R}^2$ as its domain of attraction.

(b) The feedback function U in part (a) is not bounded on $\mathbb{R}^2$. (That is, $|U(x_1, x_2)|$ can be made arbitrarily large by increasing $|(x_1, x_2)|$.) This is undesirable in practice. Show that for each constant $p > 0$, the feedback function

$$W(x_1, x_2) = -p\, f(x_1 + x_2), \qquad \text{where} \qquad f(r) = \frac{r}{\sqrt{1 + r^2}},$$

resolves this issue, by proving that

(i) the function W is bounded on $\mathbb{R}^2$;

(ii) specifying $u(t) = W(x_1(t), x_2(t))$ in $(*)$ produces a system for which the origin is an asymptotically stable equilibrium point which has all of $\mathbb{R}^2$ as its domain of attraction. [Hints: Notice that $f'(r) > 0$ $\forall r \in \mathbb{R}$. Consider

$$V(x_1, x_2) = pF(x_1) + pF(x_1 + x_2) + x_2^2,$$

where F is defined by $F'(r) = f(r)$, $F(0) = 0$.]

(c) Hold a competition between the feedback laws U and W above (taking $p = 1$), using the initial point $(x_1(0), x_2(0)) = (5, 5)$. By whatever combination of hand calculation and computer work you prefer, find (to three significant digits)

(i) the first time after which the system state $\mathbf{x}(t)$ remains forever inside the disk defined by $|\mathbf{x}| \leq 10^{-1}$,

(ii) the RMS control effort, defined by

$$\|u\|_2 = \left[\int_0^\infty |u(t)|^2\, dt\right]^{1/2}.$$

6. Consider the controlled nonlinear equation below, assuming $p \in \{1, 3, 5, 7, \dots\}$ and $\varepsilon > 0$:

$$\ddot{x} - \varepsilon \dot{x}^p + x = u, \qquad |u| \leq 1.$$

Construct a feedback control function $u(x, v)$ such that $|u(x, v)| \leq 1$ for all $(x, v) \in \mathbb{R}^2$ and the origin is a stable equilibrium point for

$$\ddot{x} - \varepsilon \dot{x}^p + x = u(x, \dot{x}).$$

Use Liapunov's theory to estimate the region of asymptotic stability.