Math 403 Problem Set 8

Due in class on Wednesday 31 March 2010

1. Let X be a real vector space. A scalar-valued function $f\colon X \to \mathbb{R}$ is called *convex* exactly when

$$\forall x_0, x_1 \in X,\ \forall t \in (0,1), \qquad f\left(tx_1 + (1-t)x_0\right) \le tf(x_1) + (1-t)f(x_0).$$

(i) Prove that $f\colon X \to \mathbb{R}$ is a convex *function* if and only if its "epigraph", namely,

$$\operatorname{epi} f \stackrel{\text{def}}{=} \{(x,r) \in X \times \mathbb{R} \,:\, r \ge f(x)\},$$

is a convex *set*.

(ii) Prove that if f is a convex function and x_0 is a point where f is differentiable, then

$$\forall x \in X, \qquad f(x) \ge f(x_0) + Df(x_0)[x - x_0].$$

Here $Df(x_0)\colon X \to \mathbb{R}$ is the derivative of f at x_0. Recall the chain rule, which says that the derivative at $t = 0$ of the scalar function $\phi(t) = f(x_0 + tv)$ is given by $Df(x_0)[v]$. In the special case of $X = \mathbb{R}^n$,

$$Df(x_0)[v] = \langle \nabla f(x_0)\,,\, v\rangle = \nabla f(x_0) \bullet v.$$

(iii) Suppose S is a convex subset of $\mathbb{R}^n$ and $x_0 \in S$ is a point of differentiability for a convex function $f\colon \mathbb{R}^n \to \mathbb{R}$. Prove:

If $-\nabla f(x_0) \in N_S(x_0)$, then x_0 minimizes f over S.

(iv) Suppose $L = L(x,v)$ is convex on the set $\mathbb{R}^n \times \mathbb{R}^n$. Suppose $\widehat{x}\colon [0,1] \to \mathbb{R}^n$ is an arc satisfying the Euler-Lagrange equation, namely,

$$\frac{d}{dt} L_v(\widehat{x}(t), \dot{\widehat{x}}(t)) = L_x(\widehat{x}(t), \dot{\widehat{x}}(t)), \qquad \text{a.e. } t \in [0,1]. \tag{EL}$$

Prove that any state trajectory $y\colon [0,1] \to \mathbb{R}^n$ satisfying $y(0) = \widehat{x}(0)$ and $y(1) = \widehat{x}(1)$ will have

$$\int_0^1 L(y(t), \dot{y}(t))\,dt \ge \int_0^1 L(\widehat{x}(t), \dot{\widehat{x}}(t))\,dt.$$

Hint: Define $p(t) = L_v(\widehat{x}(t), \dot{\widehat{x}}(t))$ to express (EL) as $(\dot{p}(t), p(t)) = \nabla L(\widehat{x}(t), \dot{\widehat{x}}(t))$.

2. A jet aircraft flying in a vertical plane has an engine which produces constant thrust; the pilot can change the jet's orientation θ to direct the thrust along any unit vector $(\cos\theta, \sin\theta)$. If the aircraft's horizontal and vertical position are denoted by x_1, x_2, and their rates of change by x_3, x_4, then the dynamics are

$$\dot{x}_1 = x_3, \quad \dot{x}_2 = x_4, \quad \dot{x}_3 = \cos\theta, \quad \dot{x}_4 = \sin\theta - g,$$

where g is the (constant) force per unit mass due to gravity. Assume that the aircraft is flying horizontally with a known velocity when $t = 0$:

$$x_1(0) = 0, \quad x_2(0) = 0, \quad x_3(0) = v_0 > 0, \quad x_4(0) = 0.$$

(a) Given smooth functions $\ell(T, \mathbf{x})$ (with scalar values) and $\phi(T, \mathbf{x})$ (with vector values), consider the problem of choosing the final time $T > 0$ and the thrust angle $\theta\colon [0,T] \to [-\pi, \pi)$ to minimize $\ell(T, \mathbf{x}(T))$ subject to $\phi(T, \mathbf{x}(T)) = 0$. Show that in the normal case, the best choice must obey the "bilinear tangent law" below for some constants m, b, n, and c:

$$\tan\theta(t) = \frac{mt + b}{nt + c}. \tag{$*$}$$

(b) Interpret this problem when $\ell(T, \mathbf{x}) = -x_3$ and $\phi(T, \mathbf{x}) = (T - N, x_2 - 1, x_4)$. Take $g = 0.100$ and evaluate the four coefficients in $(*)$ numerically for the four cases $N = 8.00, 6.00, 4.00, 2.00$.

3. The problem of stopping a controlled harmonic oscillator in prescribed time T with minimum energy can be expressed succinctly as follows:

$$\begin{aligned}
\text{minimize} \quad & J[u] \stackrel{\text{def}}{=} \int_0^T \tfrac{1}{2}|u(t)|^2\,dt, \\
\text{over all} \quad & u\colon [0,T] \to \mathbb{R} \text{ piecewise continuous} \\
\text{subject to} \quad & \ddot{y}(t) + y(t) = u(t), \ \text{ a.e. } t \in [0,T], \\
& y(0) = y_0, \ \dot{y}(0) = v_0, \\
& y(T) = 0, \ \dot{y}(T) = 0.
\end{aligned}$$

Apply the Maximum Principle to identify an extremal control.

Preview: You will find a linear combination of sinusoids for $\widehat{u}$. The coefficients in this combination will be determined by the problem data y_0, v_0, and T. The explicit form of the coefficients is a little nasty, so don't work it out. Just present a 2×2 system of linear equations they must satisfy and assume the reader can solve it.

4. Find an explicit formula for the minimizing control u in this fixed-endpoint problem:

$$\begin{aligned}
\text{minimize} \quad & \int_0^1 u(t)^2\,dt & & \\
\text{subject to} \quad & \dot{x} = y, & & x(0) = 1, \quad x(1) = 0, \\
& \dot{y} = x + u, & & y(0) = 0, \quad y(1) = 0.
\end{aligned}$$