Math 403 Problem Set 7

Due in class on Wednesday 24 March 2010

1. Solve the following problem with $a = 0$, then repeat with $a = 1$:

$$\begin{aligned} \text{minimize} \quad & J = \int_0^2 (2x - 3u - \frac{a}{2}u^2)\,dt \\ \text{subject to} \quad & \dot{x} = x + u, \qquad x(0) = 5, \\ & 0 \le u \le 2. \end{aligned}$$

(*Suggestion*: When $a = 1$, sketch the graph of $w \mapsto H(t, x(t), p(t), w)$ before analyzing extremals.)

2. Find extremal control-state pairs in this problem, where $T > 0$ and $B > 0$ are fixed in advance:

$$\begin{aligned} &\text{minimize} \int_0^T x(s)\sin u(s)\,ds \\ &\text{subject to } \dot{x}(s) = \cos u(s) \text{ a.e. } s \in [0,T], \quad x(0) = 0,\ x(T) = B, \\ &\qquad\qquad \dot{y}(s) = \sin u(s) \text{ a.e. } s \in [0,T], \quad y(0) = 0,\ y(T) = 0. \end{aligned}$$

(i) Discuss the situation in which there is a nontrivial abnormal extremal.

(ii) Show that any normal extremal control produces a trajectory that is the arc of a circle in the (x, y)-plane.

(iii) Modify the problem by removing the right endpoint constraint $x(T) = B$, so that $x(T)$ is free to take any value in $(0, +\infty)$. Describe all extremals. Are there any abnormal ones?

3. Consider the following calculus of variations problem involving a velocity constraint:

$$\min \left\{ \int_0^2 (3\dot{x}(t) - 5x(t))\ dt\ :\ |\dot{x}(t) - x(t) - 1| \le 1,\ x(0) = 5 \right\}.$$

Show how the choices $u = \dot{x} - x$ and $y(t) = \displaystyle\int_0^t (3\dot{x}(r) - 5x(r))\ dr$ can be used to transform this into a fixed-time optimal control problem with two-dimensional state space. Then find the minimizing piecewise smooth function $x(\cdot)$.

4. Find extremal policies for the problem below, where $\delta \in (0, 1)$ and $\xi > 0$ are given:

$$\min \left\{ \int_0^\pi e^{-\delta t}\,(u(t) - 1)\,x(t)\,dt\ :\ \dot{x}(t) = u(t)x(t),\ x(0) = \xi,\ u(t) \in [0, 1] \right\}.$$

The cases when δ is sufficiently near 0 and when δ is sufficiently near 1 are qualitatively different. Explain why, and show how to find the δ-value where the change in behaviours takes place. Sketch the extremal state arc x and evaluate $x(\pi)$ in terms of ξ in the special case $\delta = \frac{1}{2}$.

5. Consider the following nonlinear optimal control problem whose state is scalar-valued:

$$\begin{aligned}
\text{minimize}\quad & \Lambda[u] := \int_0^1 x(t)\,[u(t)-1]\;dt \\
\text{subject to}\quad & \dot{x}(t) = (\alpha + \beta u(t))\,x(t) \text{ a.e. } t \in [0,1], \\
& 0 \le u(t) \le 1 \text{ a.e. } t \in [0,1], \\
& x(0) = \xi,\; x(1) = \Xi.
\end{aligned}$$

Here α, β, ξ, and Ξ are given real numbers. Assume that $\beta \neq 0$, $\xi \neq 0$, and that Ξ lies in the *open* interval with endpoints ξe^{α} and $\xi e^{\alpha+\beta}$.

(i) Show that if $\alpha + \beta = 0$, then every admissible control is optimal. Then restrict your attention to the case $\alpha + \beta \neq 0$.

(ii) Let u be an extremal control. Show that there is a *monotonic* function $\sigma(t)$ such that

$$u(t) \in \begin{cases} \{1\}, & \text{if } \sigma(t) > 0, \\ [0,1], & \text{if } \sigma(t) = 0, \\ \{0\}, & \text{if } \sigma(t) < 0. \end{cases}$$

(iii) Show that the problem is normal, so that the result of step (b) holds for a *strictly monotonic* function σ.

(iv) In each of the two cases $\xi(\alpha+\beta) > 0$ and $\xi(\alpha+\beta) < 0$, identify the unique extremal control and corresponding state. Sketch them in the case $\alpha = -1$, $\beta = 2$, $\xi = 1$, $\Xi = 1$.