

Math 403 Problem Set 6

Due in class on Friday 12 March 2010

1. Consider the dynamical system

$$\ddot{y}(t) = u(t), \quad 0 \leq t \leq 2; \quad y(0) = 0, \quad \dot{y}(0) = 0.$$

Call a piecewise continuous function $u: [0, 2] \rightarrow \mathbb{R}$ a “control” if it obeys

$$|u(t)| \leq 1, \quad t \in [0, 2].$$

- (a) Find the control that maximizes the quantity $\dot{y}(2) - y(2)$.
- (b) Among all controls which make $y(2) = 1$, find the one that maximizes $\dot{y}(2)$.

2. Consider the single-input, control-constrained system

$$\dot{x} = y, \quad \dot{y} = u, \quad |u| \leq 1 \quad \text{a.e. } t$$

and the target set

$$S = \{(x, y) : 0 \leq y \leq x + 1, x \leq 0\}.$$

For the problem of steering a given initial point in the (x, y) -plane to the target set S , carefully describe the set of initial points for which the minimum-time trajectory ends at a point where $y = x + 1$. Express the corresponding control strategies as functions of position in the (x, y) -plane.

3. Synthesize a feedback control strategy for minimum-time transfer to the origin, given

$$\dot{x}_1 = -3x_1 + 3u, \quad \dot{x}_2 = -x_2 + u, \quad |u| \leq 1.$$

That is, use the Maximum Principle to help draw a “map” of the (x_1, x_2) -plane that specifies one u -value for each location. Trajectories generated by the corresponding controls should reach the origin in minimum time.

4. Prove: For any two nonempty convex sets C and D in \mathbb{R}^n satisfying

$$0 \in C \cap D, \quad \text{int}(C) \neq \emptyset, \quad \text{int}(C) \cap D = \emptyset,$$

there must be a nonzero vector \mathbf{w} in \mathbb{R}^n such that

$$\mathbf{w} \bullet \mathbf{c} \leq \mathbf{w} \bullet \mathbf{d} \quad \forall \mathbf{c} \in C, \mathbf{d} \in D.$$

Clue: Recall Theorem C.2 of the online notes. Use it twice. First, to show that the set $S \stackrel{\text{def}}{=} \{\mathbf{y} - \mathbf{x} : \mathbf{y} \in D, \mathbf{x} \in C\}$ is convex, with $\mathbf{0} \in S$ and $\mathbf{0} \notin \text{int}(S)$. Second, to get the desired conclusion.