Math 403 Problem Set 4

Due in class on Friday 5 February 2010

Problems 1–3 can probably be done by hand. Problem 4 really needs a computer to help; problem 5 is can go either way. If and when you use the computer, hand in enough documentation of what you did to make it possible for the reader to reproduce your results easily.

1. For a matrix pair (A, B), with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, prove that the following are equivalent:

(a) (A, B) is controllable

(b) for each $F \in \mathbb{R}^{m\times n}$, $(A + BF, B)$ is controllable.

(The point: Applying a controller of the form $u = Fx + \widetilde{u}$ leads from $\dot{x} = Ax + Bu$ to $\dot{x} = (A + BF)x + B\widetilde{u}$. Hence linear feedback can be used to give the second system better eigenvalues, without sacrificing the freedom to control it completely using the remaining input $\widetilde{u}$.)

2. Consider this linear control system, in which α, β are constant parameters:

$$(1) \qquad \begin{cases} \dot{x}_1 = \phantom{\alpha x_1 + {}} x_2 + u \\ \dot{x}_2 = \alpha x_1 + \beta x_2 \end{cases}$$

(a) Under what conditions on the constants α, β is system (1) controllable?

(b) Suppose α, β are chosen so that system (1) **is not controllable**. Completely describe the attainable sets $\mathcal{A}(t; \mathbf{0})$ for $t > 0$.

(c) Now suppose α, β are chosen so system (1) **is controllable**. Find constants c and k so that system (1) is equivalent to the following system in canonical form

$$(2) \qquad \begin{cases} \dot{y}_1 = \phantom{cy_1 + {}} y_2 \\ \dot{y}_2 = cy_1 + ky_2 + u. \end{cases}$$

Carefully explain the transformation relating the variables $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$.

3. Consider the 2×2 system with matrices

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(a) Show how to choose the feedback matrix $F = [\, f_0 \quad f_1 \,]$ so that the eigenvalues of $A + BF$ are λ_1, λ_2. (That is, give formulas expressing f_0, f_1 in terms of $a_0, a_1, \lambda_1, \lambda_2$.)

(b) Evaluate F explicitly when $a_0 = 0$, $a_1 = 70$, $\lambda_1 = -50(1 + i)/\sqrt{2}$, and $\lambda_2 = \overline{\lambda_1}$.

(c) Again taking $a_0 = 0$, $a_1 = 70$, suppose design constraints require that $|f_0| \leq 1$ and $|f_1| \leq 1$. Choose F to minimize the quantity

$$\max \{\Re\mathrm{e}(\lambda_1), \Re\mathrm{e}(\lambda_2)\}.$$

4. Linearizing the famous "broom-balancing problem" leads to a system $\dot{\mathbf{x}} = A\mathbf{x} + Bu$ with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}. \tag{$*$}$$

(a) Show that system $(*)$ is controllable.

(b) Find a coordinate-transformation matrix P that puts $(*)$ into controllable canonical form.

(c) Find a feedback matrix F, for which all four eigenvalues of $A + BF$ equal -1.

(d) Show that if $y = x_3$ is the only state component we can measure, then $(*)$ is not observable. Give an example of two 0-input state trajectories that cannot be distinguished using this measurement; give a common-sense explanation based on the physical apparatus.

(e) Show that if $y = x_1$ is the only state component we can measure, then $(*)$ is observable.

(f) Using $y = x_1$ as the output, design an output-feedback controller that stabilizes the system. (Arrange for all eigenvalues of the error dynamics to equal -5.)

(g) Compare the output-feedback controller to the state-feedback controller numerically as follows. Plot, on the same axes, the function $t \mapsto x_1(t)$ corresponding to initial point $\mathbf{x}(0) = (1, 0, 0, 0)$ obtained using the 4×4 state-feedback system in (c) and the same function generated using the 8×8 output-feedback system in (h). Do the same for the the function $t \mapsto x_3(t)$. Then repeat this procedure (plotting both x_1 and x_3) for the initial point $\mathbf{x}(0) = (0, 0, 1, 0)$. In both cases, take the initial state estimate as $\mathbf{z}(0) = \mathbf{0}$.

5. Consider the linearized satellite system with only tangential thrust. Taking $\rho = \omega = 1$, this is described by the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Construct a feedback matrix F such that $A + BF$ has eigenvalues -2, -1, and $-\frac{1}{2} \pm \frac{1}{2}i$.