Math 403 Problem Set 3

Due in class on Wednesday 27 January 2010

1. A linear system with $x \in \mathbb{R}^3$ and $u \in \mathbb{R}$ is described by the state equation

$$\dot{x}(t) = \begin{bmatrix} 0 & 5 & -2 \\ 1 & 0 & 0 \\ 0 & 2 & -2 \end{bmatrix} x(t) + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} u(t).$$

(a) Is this system controllable?

(b) Carefully describe the attainable set $\mathcal{A}(1; \mathbf{0})$. (The values of u are unrestricted.)

(c) Suppose that state variable feedback is introduced, so that

$$u(t) = [\,\alpha \quad 0 \quad 0\,]\, x(t)$$

for some constant α. Is it possible to choose α to force all of the eigenvalues to the origin? If not, how many of the eigenvalues can be moved to the origin, and where are the others under these conditions?

2. Consider the linear system

$$\dot{x}(t) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} u(t).$$

(a) Is the system controllable?

(b) Compute the zero-input response for $x(0) = \xi = (\xi_1, \xi_2, \xi_3)$.

(c) Give a complete description of the attainable set $\mathcal{A}(1; (1, 0, 1))$.

3. If the constants a_1, a_2, …, a_n are given, prove that for any ξ, $\eta \in \mathbb{R}$ there exists a piecewise continuous control function $u : [0, 1] \to \mathbb{R}$ such that the solution $x(t)$ to the scalar initial-value problem

$$x^{(n)}(t) + a_1 x^{(n-1)}(t) + \cdots + a_{n-1}\dot{x}(t) + a_n x(t) = u(t), \qquad x(0) = \xi$$

satisfies $x(1) = \eta$.

4. Consider the linear initial-value problem $\dot{x}(t) = Ax(t)$, $x(0) = \xi$, in which A is a real $n \times n$ matrix. For each of the following statements, give a proof or a counterexample. (You may refer to theorems discussed in class.)

(a) If every choice of ξ gives rise to a solution for which $\sup_{t>0} |x(t)|$ is finite, then every eigenvalue λ of A satisfies $\Re\lambda \leq 0$.

(b) If every eigenvalue λ of A satisfies $\Re\lambda \leq 0$, then every choice of ξ gives rise to a solution for which $\sup_{t>0} |x(t)|$ is finite.

5. Consider this control system with a scalar input u:

$$(*) \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} u.$$

(i) Find all values of a and b for which system $(*)$ is controllable.

(ii) Assuming $b \neq 0$, find all vectors (v_1, v_2) such that using the feedback law $u = \mathbf{v} \bullet \mathbf{x}$ in $(*)$ produces a system in which every trajectory obeys $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$. Give a general solution in terms of a and b, then sketch the appropriate region of the (v_1, v_2)-plane for the special cases $a = 2$, $b = 1$.

6. Given a real matrix $A \in \mathbb{R}^{n \times n}$, let $t \mapsto \mathbf{x}(t;\xi)$ denote the unique solution of

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t), \ t > 0; \qquad \mathbf{x}(0) = \xi.$$

Sometimes it is impossible to measure all n components of the state vector directly, and one has access only to the p components of an "observation" $\mathbf{y} \overset{\text{def}}{=} C\mathbf{x}$, where C is a given $p \times n$ matrix. Let $\mathbf{y}(t;\xi) = C\mathbf{x}(t;\xi)$.

Prove that the following are equivalent:

(a) Identical observation histories guarantee identical initial states, i.e.,

$$\Big[\text{ for some } T > 0, \text{ one has } \mathbf{y}(t;\xi) = \mathbf{y}(t;\eta) \quad \forall t \in (0,T) \Big] \qquad \Longrightarrow \qquad \xi = \eta.$$

(b) Different initial states will cause differences in observation values after arbitrarily short times, i.e.,

$$\xi \neq \eta \qquad \Longrightarrow \qquad \Big[\text{ for each } T > 0, \text{ there is some } t \in (0,T) \text{ such that } \mathbf{y}(t;\xi) \neq \mathbf{y}(t;\eta) \Big]$$

(c) The matrix pair (A^T, C^T) is controllable.

Statements (a) and (b) explain why the matrix pair (A, C) is called *observable* in situations (a)–(c).

7. Consider the forced harmonic oscillator, for which the governing equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

We want to drive this system from the initial state $\xi = (1, 0)$ to the origin at time $t = 2\pi$.

(a) Is this transfer possible using some piecewise continuous function u?

(b) Is it possible using a piecewise *constant* control function of the form shown below?

$$u(t) = \begin{cases} u_1, & \text{if } 0 \le t < 2\pi/3, \\ u_2, & \text{if } 2\pi/3 \le t < 4\pi/3, \\ u_3, & \text{if } 4\pi/3 < t \le 2\pi. \end{cases}$$

If so, find all triples (u_1, u_2, u_3) that do the job.