Math 403 Problem Set 2

Due in class on Wednesday 20 January 2010

1. Find a 2×2 matrix-valued function $C(\cdot)$ with these properties:

$$\ddot{C}(t) + \begin{bmatrix} 13 & -5 \\ -5 & 13 \end{bmatrix} C(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ t \in \mathbb{R}; \qquad C(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \dot{C}(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hint: Look for a matrix W that puts the system into the form $\ddot{C} + W^2 C = 0$, $C(0) = I$, $\dot{C}(0) = 0$. The analogous scalar equation has a known solution; guess and check an analogous matrix-valued solution.

2. Consider the time-varying matrix below, where α and π are positive constants:

$$A(t) = \begin{bmatrix} 0 & \alpha(\pi - t)^{-2} \\ 0 & 0 \end{bmatrix}.$$

(a) Find the fundamental matrix $\Phi = \Phi(t, \tau)$.

(b) Write the unique solution for the vector initial-value problem

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t), \qquad \mathbf{x}(2\pi) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

State the interval of validity for this solution.

(c) Simplify the matrix-valued expression $\exp\left(\int_\tau^t A(r)\,dr\right)$ and compare the result with $\Phi(t, \tau)$.

3. (a) Prove that for matrix-valued functions A and B of appropriate dimensions,

$$\frac{d}{dt}\left(A(t)B(t)\right) = \frac{dA}{dt}B + A\frac{dB}{dt}.$$

(b) Find a 2×2 matrix function M for which $\frac{d}{dt}\left[M(t)^2\right] \neq 2M(t)\dot{M}(t)$.

(c) Consider this initial-value problem involving a time-varying $n \times n$ matrix A:

$$\dot{x}(t) = A(t)x(t), \qquad x(0) = \xi. \tag{1}$$

If $n = 1$ or if A is a constant matrix, the unique solution of (1) is known:

$$x(t) = \exp\left(\int_0^t A(r)\,dr\right)\xi. \tag{2}$$

Show that, in spite of this, formula (2) is not universal: display a specific matrix-valued function $A(\cdot)$ for which formula (2) defines an arc x that *disobeys* (1).

4. The "Euler equations" describe the rotational motion of a rigid body free from external forces and torques:

$$\dot{\omega}_1 = \omega_2\omega_3\left(\frac{I_2 - I_3}{I_1}\right), \quad \dot{\omega}_2 = \omega_3\omega_1\left(\frac{I_3 - I_1}{I_2}\right), \quad \dot{\omega}_3 = \omega_1\omega_2\left(\frac{I_1 - I_2}{I_3}\right).$$

Here $\omega_1(t)$, $\omega_2(t)$, and $\omega_3(t)$ are the components of angular velocity and I_1, I_2, and I_3 are the (constant) moments of inertia about the principal axes. In this question, we assume that $I_1 > I_2 > I_3 > 0$.

(a) Find all nonzero equilibrium points of this system. Derive a linearized model describing the motion of the system in a neighbourhood of each equilibrium point.

(b) Show that the equilibrium solution $(\overline{\omega}_1(t), \overline{\omega}_2(t), \overline{\omega}_3(t)) = (0, \Omega, 0)$, $\Omega \neq 0$, is unstable. Comment on the stability of the other equilibrium points.

5. The famous "broom-balancing problem" involves a cart of mass M running along a straight horizontal track under the influence of a force u. Friction proportional to the cart's velocity opposes its motion; the friction coefficient is k. On top of the cart is a frictionless hinge connected to a lightweight rod of length ℓ, with a concentrated mass m at its free end. Writing δ for the cart's horizontal displacement from the origin and ϕ for the angle between the rod and the vertical leads to this system of equations:

$$\begin{aligned}
&(M+m)\ddot{\delta} + m\ell\ddot{\phi}\cos\phi - m\ell\dot{\phi}^2\sin\phi + k\dot{\delta} = u,\\
&\ell\ddot{\phi} - g\sin\phi + \ddot{\delta}\cos\phi = 0.
\end{aligned}$$

(Here $g > 0$ is a constant representing the acceleration due to gravity.) In the following steps, assume $M, m, k, \ell, g = 1$ for simplicity.

(a) Let $\mathbf{x} = (\delta, \dot{\delta}, \phi, \dot{\phi})$. Find a (nonlinear) system of controlled differential equations for $\mathbf{x}$.

(b) Show that the vector $\mathbf{x} = \mathbf{0}$, which represents the cart parked at the origin with the rod perfectly vertical and stationary, is an equilibrium state.

(c) Linearize the system in (a) about the equilibrium point in (b). The desired outcome is a *linear* system

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu \qquad (*)$$

closely related to the one in part (a). Use the same variable names ($\mathbf{x}, \delta, \phi$, etc.) as in (a), even though the new system is only an approximation of the original one.

(d) Find the eigenvalues of the linearized system matrix A and comment on the stability of $(*)$ when $u \equiv 0$.