Math 403 Problem Set 1

Due in class on Wednesday 13 January 2010

This week's problems are meant to be solved by hand calculation. But there is no penalty for using a computer to check your work after the fact, or even to suggest the correct answer before you start to pursue it.

1. Consider this system of differential equations involving a constant k:

$$(*) \quad \begin{cases} \dot{x}_1 = -x_1 - x_2, \\ \dot{x}_2 = -x_1 - kx_2 . \end{cases}$$

Let A denote the 2×2 matrix for which $(*)$ has the form $\dot{\mathbf{x}} = A\mathbf{x}$ with $\mathbf{x} = (x_1, x_2)$.

(i) Find the eigenvalues of A and the general solution of $(*)$ when $k = \frac{1}{2}$.

(ii) Repeat part (i), but take $k = 2$.

(iii) Compare and contrast the qualitative behaviour of $(*)$ in the two cases above. Determine the value of k in the interval $[\frac{1}{2}, 2]$ where the transition between these two types of behaviour occurs. (This k-value is called a *bifurcation point.*)

2. Evaluate e^{At} by hand for the two choices of A shown below.

$$\text{(i)} \quad A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}; \qquad \text{(ii)} \quad A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Hint: In both cases, splitting $A = (A - rI) + (rI)$ and working out $e^{(A-rI)t}$ is a good first step. A smart choice of r is the key to success.

3. Find e^{At} for the matrix A defined below. (Here ω and R are positive constants.)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -\omega^2/R \\ 0 & 1/R & 0 \end{bmatrix}.$$

4. Consider this controlled linear system with $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $u \in \mathbb{R}$:

$$(*) \quad \dot{\mathbf{x}} = A\mathbf{x} + Bu, \qquad \text{where} \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(a) Find e^{At}.

(b) Assuming $\mathbf{x}(0) = \xi = (\xi_1, \xi_2)$, write an integral formula for $\mathbf{x}(t)$ in terms of ξ and $u(\cdot)$. Then extract a separate scalar formula for each component of $\mathbf{x}(t)$.

(c) Find a piecewise continuous function $u\colon [0,1] \to \mathbb{R}$ such that the solution of $(*)$ with $\mathbf{x}(0) = (-1, 3)$ obeys $\mathbf{x}(1) = (0, 0)$. [Many correct answers exist; some of the simplest have discontinuities.]

(d) Find a constant 1×2 matrix $F = [f_1 \ f_2]$ for which the choice $u = F\mathbf{x}$ transforms $(*)$ into a system

$$\dot{\mathbf{x}} = (A + BF)\mathbf{x}$$

in which every trajectory converges to 0 as $t \to \infty$.