

C. Convexity

C.1. Definition. Let X be a real vector space. A subset S of X is called *convex* if, for every pair of points s_0 and s_1 in S , one has

$$s_\alpha \stackrel{\text{def}}{=} (1 - \alpha)s_0 + \alpha s_1 \in S \quad \forall \alpha \in (0, 1).$$

C.2. Theorem. Let $S \subseteq \mathbb{R}^n$ be a convex set. Then

- (a) If $s_0 \in \text{int } S$ and $s_1 \in \text{cl } S$, then $(1 - \alpha)s_0 + \alpha s_1 \in \text{int } S$ for all $\alpha \in (0, 1)$;
- (b) $\text{int } S$ is convex; $\text{int } S = \text{int}(\text{cl } S)$; and if $\text{int } S \neq \emptyset$, then $\text{cl } S = \text{cl}(\text{int } S)$;
- (c) a point x lies outside $\text{cl } S$ if and only if there exists a nonzero vector v in \mathbb{R}^n such that

$$\langle v, s \rangle \leq \langle v, x \rangle - 1 \quad \forall s \in S.$$

- (d) a point x lies outside $\text{int } S$ if and only if there exists a nonzero vector v in \mathbb{R}^n such that

$$\langle v, s \rangle \leq \langle v, x \rangle \quad \forall s \in S.$$

Proof [(c) and (d) only]. “(c) \Rightarrow ” Suppose \hat{x} is a point outside $\text{cl } S$. Consider the problem of minimum distance from \hat{x} to the set $\text{cl } S$:

$$\min \{ |s - \hat{x}| : s \in \text{cl } S \}.$$

Since the constraint set $\text{cl } S$ is closed and the objective function $s \mapsto |s - \hat{x}|$ is continuous, with compact sublevel sets, a minimizing s -value certainly exists: call it \hat{s} . (\hat{s} is “a nearest point” in $\text{cl } S$ to \hat{x} .) Observe that for any point s' in S ,

$$\begin{aligned} |\hat{s} - \hat{x}|^2 &\leq |s' - \hat{x}|^2 = |s' - \hat{s}|^2 + 2\langle s' - \hat{s}, \hat{s} - \hat{x} \rangle + |\hat{s} - \hat{x}|^2 \\ &\iff \langle \hat{x} - \hat{s}, s' - \hat{s} \rangle \leq \frac{1}{2}|s' - \hat{s}|^2. \end{aligned} \quad (*)$$

Now for any fixed s in S , we can substitute $s' = \hat{s} + \alpha(s - \hat{s})$ in (*) for any $\alpha \in (0, 1)$, thanks to the convexity of the set S . This gives

$$\langle \hat{x} - \hat{s}, \alpha(s - \hat{s}) \rangle \leq \frac{1}{2}\alpha^2|s - \hat{s}|^2 \quad \forall \alpha \in (0, 1).$$

Dividing the result by α and taking the limit as $\alpha \rightarrow 0^+$, we find

$$\langle \hat{x} - \hat{s}, s - \hat{s} \rangle \leq 0.$$

This works for any $s \in S$, and it implies that the vector $v_0 = \hat{x} - \hat{s}$ satisfies

$$\begin{aligned} \langle v_0, s - \hat{x} \rangle &= \langle v_0, s - \hat{s} \rangle + \langle v_0, \hat{s} - \hat{x} \rangle \\ &= \langle v_0, s - \hat{s} \rangle - |v_0|^2 \leq 0 - |v_0|^2. \end{aligned}$$

In particular, the vector $\hat{v} = v_0/|v_0|^2$ obeys

$$\langle \hat{v}, s - \hat{x} \rangle \leq -1,$$

which gives the desired result. (It is important to think about why the vector \hat{v} is well-defined, i.e., why $v_0 \neq 0$, and to recognize that \hat{v} may well fail to be a unit vector.)

“(c) \Leftarrow ” Suppose x is a point for which some vector v satisfies $\langle v, s - x \rangle \leq -1$ for all s in S . If x were actually a point in $\text{cl } S$, then there would be a sequence of points s_k in S converging to x . Every point in this sequence would satisfy $\langle v, s_k - x \rangle \leq -1$, and taking the limit as $k \rightarrow \infty$ would produce $0 \leq -1$. This cannot be: thus x must lie outside $\text{cl } S$.

“(d) \Rightarrow ” Consider now the case where the given point \hat{x} lies outside $\text{int } S$. This is more delicate, because it allows for the possibility that \hat{x} is a boundary point of S . It is here that part (b) comes into play: since S is convex, knowing $\hat{x} \notin \text{int } S$ implies that in fact $\hat{x} \notin \text{int}(\text{cl } S)$. Therefore \hat{x} is realizable as a limit of a sequence of points x_k lying outside $\text{cl } S$. For each k , we may apply part (c) to find a vector v_k such that

$$\langle v_k, s - x_k \rangle \leq -1 \quad \forall s \in S, \forall k.$$

In particular, if we set $\hat{v}_k = v_k/|v_k|$, we obtain a sequence of unit vectors in \mathbb{R}^n , each satisfying

$$\langle \hat{v}_k, s - x_k \rangle \leq 0 \quad \forall s \in S. \quad (**)$$

Along a suitable subsequence, we have $\hat{v}_k \rightarrow \hat{v}$ for some unit vector \hat{v} ; taking the limit in (**) above gives the desired result:

$$\langle \hat{v}, s - \hat{x} \rangle \leq 0 \quad \forall s \in S.$$

“(d) \Leftarrow ” Suppose x is a point for which some vector v satisfies $\langle v, s - x \rangle \leq 0$ for all s in S . If x were actually a point in $\text{int } S$, then there would be some positive radius $\rho > 0$ such that the open ball $\mathbb{B}(x; \rho)$ is a subset of S . In particular, for every unit vector u in \mathbb{R}^n , the point $s = x + \frac{1}{2}\rho u$ would lie in S , and the inequality above would give $\langle v, u \rangle \leq 0$. This certainly implies $v = 0$. So if the indicated inequality holds for $v \neq 0$, then it must be true that $x \notin \text{int } S$. /////

The situation in parts (c) and (d) of the previous result is so important that we define the terms formally.

C.3. Definition. Given a convex set S , a point $s \in \text{cl } S$, and a vector v , we say that v is an outward normal to S at s , and write $v \perp S$ at \hat{s} , exactly when

$$\langle \hat{v}, x - s \rangle \leq 0 \quad \forall x \in S.$$

The set of all outward normals at s is written $N_S(s)$.

The point of the theory just developed is that for convex sets, boundary points have an easily accessible geometric characterization: a boundary point is one where there exists at least one nonzero outward normal to the set, and the outward normal is described by a simple inequality.

D. Attainable Sets and Boundary Trajectories

Now a new ingredient enters our problem setup: explicit constraints on the control values. Suppose the values of the control input are all required to lie in some preassigned subset U of \mathbb{R}^m . Then the dynamics become

$$\dot{x}(t) = Ax(t) + Bu(t), \quad u(t) \in U, \quad \text{a.e.} \quad (*)$$

The attainable set will be influenced by this new constraint, so we adapt our notation to express this: for each $t > 0$ and $\xi \in \mathbb{R}^n$, we write

$$\mathcal{A}(t; \xi, U) := \left\{ e^{At}\xi + \int_0^t e^{A(t-s)}Bu(s) ds : u(s) \in U \forall s \in [0, t] \right\}.$$

To put the notions of convexity to work in our control problem, we assume that the set of allowed input values U is convex. As an instance of the general rule that the affine image of a convex set is again a convex set, we obtain the following result:

D.0. Proposition. *If the control set $U \subseteq \mathbb{R}^m$ is convex, then each attainable set $\mathcal{A}(t; \xi, U)$ ($t > 0$, $\xi \in \mathbb{R}^n$) is convex.*

Proof. Let $\eta_0, \eta_1 \in \mathcal{A}(t; \xi, U)$. This means that there exist $u_0, u_1 \in PWC([0, t]; \mathbb{R}^n)$ such that

$$\begin{aligned} \eta_0 &= e^{At}\xi + \int_0^t e^{A(t-r)}Bu_0(r) dr, \\ \eta_1 &= e^{At}\xi + \int_0^t e^{A(t-r)}Bu_1(r) dr. \end{aligned}$$

For any $\alpha \in (0, 1)$, it follows that

$$\eta_\alpha := (1 - \alpha)\eta_0 + \alpha\eta_1 = e^{At}\xi + \int_0^t e^{A(t-r)}Bu_\alpha(r) dr,$$

where $u_\alpha(t) := (1 - \alpha)u_0(t) + \alpha u_1(t)$. Since U is convex, one has $u_\alpha \in PWC([0, t]; U)$, so the last expression shows that $\eta_\alpha \in \mathcal{A}(t; \xi, U)$, as required. ////

Boundary Trajectories. When the attainable set $\mathcal{A}(T; \xi, U)$ is convex, we can get a good idea of its shape by looking at the boundary. Once we know bdy $\mathcal{A}(T; \xi, U)$, we can recover at least the closure of the set $\mathcal{A}(T; \xi, U)$ simply by including all points inside the boundary (“taking the convex hull”).

As a first attempt to characterize the boundary of $\mathcal{A}(T; \xi, U)$, we look at those points of the boundary that actually belong to the set $\mathcal{A}(T; \xi, U)$ —i.e., those boundary points of the form $x(T)$, where x is the state response starting from ξ associated with some piecewise continuous control taking values in U .

D.1. Lemma. Suppose U is convex, and (\hat{u}, x) is a control-state pair for $(*)$ on $[0, T]$. If $x(T) \in \text{bdy } \mathcal{A}(T, x(0), U)$, then the following statements about a vector $w \in \mathbb{R}^n$ are equivalent:

- (i) $w \in N_{\mathcal{A}(T, x(0), U)}(x(T))$,
- (ii) $\int_0^T \langle e^{A^T(T-r)} w, B[u(r) - \hat{u}(r)] \rangle dr \leq 0$ for all $u \in PWC([0, T], U)$,
- (iii) for almost every $r \in [0, T]$, one has

$$\langle e^{A^T(T-r)} w, B[v - \hat{u}(r)] \rangle \leq 0 \quad \text{for all } v \in U.$$

Proof. (i \Leftrightarrow ii) By definition of the outward normal, statement (i) is equivalent to

$$\langle w, \eta - x(T) \rangle \leq 0 \quad \forall \eta \in \mathcal{A}(T; x(0), U).$$

In detail, this means that every $u \in PWC([0, T]; U)$ satisfies

$$\begin{aligned} & \left\langle w, \left(e^{AT} x(0) + \int_0^T e^{A(T-r)} B u(r) dr \right) - \left(e^{AT} x(0) + \int_0^T e^{A(T-r)} B \hat{u}(r) dr \right) \right\rangle \leq 0 \\ & \iff \int_0^T \langle w, e^{A(T-r)} B (u(r) - \hat{u}(r)) \rangle dr \leq 0 \\ & \iff \int_0^T \langle e^{A^T(T-r)} w, B (u(r) - \hat{u}(r)) \rangle dr \leq 0. \end{aligned}$$

Hence (ii) holds. But the argument is completely reversible.

(ii \Rightarrow iii) Pick a time $\theta \in [0, T)$ and a point $v \in U$ arbitrarily. For each fixed $h \in (0, T - \theta)$, define the control

$$u_h(t) = \begin{cases} v, & \text{if } \theta < t < \theta + h, \\ \hat{u}(t), & \text{otherwise.} \end{cases}$$

Each $u_h(\cdot)$ so defined belongs to $PWC([0, T], U)$, so must confirm statement (ii). Hence, dividing by $h > 0$, we find

$$\frac{1}{h} \int_{\theta}^{\theta+h} \langle e^{A^T(T-r)} w, B(v - \hat{u}(r)) \rangle dr \leq 0.$$

Since $\hat{u}(\cdot)$ is piecewise continuous by hypothesis, its one-sided limit from the right at time θ exists. So in the limit as $h \rightarrow 0^+$ above, the fundamental theorem of calculus gives

$$\langle e^{A^T(T-\theta)} w, B(v - \hat{u}(\theta^+)) \rangle \leq 0.$$

This development works for any $v \in U$ and $\theta \in [0, T)$. Since $u(\theta^+) = u(\theta)$ holds for all $\theta \in [0, T)$ with at most finitely many exceptions, statement (iii) follows. (Just write $r = \theta$.)

(iii \Rightarrow ii) Obvious. ////

Controls that drive the state vector to the boundary of the attainable set turn out to satisfy a certain system of equations shown in Definition D.2 below. The definition relies on the **pre-Hamiltonian** function $H: \mathbb{R}^n \times \mathbb{R}^n \times U \rightarrow \mathbb{R}$ defined by

$$H(x, p, u) := p^T(Ax + Bu) = p \bullet (Ax + Bu).$$

The “Hamiltonian” part of this function’s name comes from the analogy between conditions (a)–(b) below and the Hamiltonian equations of motion in classical mechanics. The prefix “pre-” arises because the analogy with classical mechanics gets even better when one considers the “true Hamiltonian” $\mathcal{H}(x, p) := \max_{v \in U} H(x, p, v)$ —a figure in more advanced courses on this subject.

D.2. Definition. A (control,state)-pair $(\hat{u}(\cdot), x(\cdot))$ defined on the interval $[a, b]$ is called *extremal* if there exists a piecewise smooth function $p: [a, b] \rightarrow \mathbb{R}^n$ obeying these four conditions:

- (a) the adjoint equation, $-\dot{p}(t) = \nabla_x H(x(t), p(t), \hat{u}(t)) = A^T p(t)$ a.e.,
- (b) the state equation, $\dot{x}(t) = \nabla_p H(x(t), p(t), \hat{u}(t)) = Ax(t) + B\hat{u}(t)$ a.e.,
- (c) the maximum condition, $H(x(t), p(t), \hat{u}(t)) \geq H(x(t), p(t), v)$ for all $v \in U$, a.e.,
- (d) the nontriviality condition, $|p(t)| \neq 0 \forall t \in [a, b]$.

(Here the abbreviation “a.e.” stands for “at all but finitely many points of $[0, T]$ ”.)[†]

Remarks. 0. Line (b) simply restates the differential equation in (*), so it contains no new information about an admissible (control,state) pair. However, it does combine with line (a) to produce a system of $2n$ differential equations for the $2n$ components of $x(\cdot)$ and $p(\cdot)$ that looks a lot like the “Hamiltonian systems” loved by experts in dynamics around the world.

1. If conditions (a)–(d) hold for a function p , then they also hold for the function αp for any scalar $\alpha > 0$. This is occasionally useful . . . one can simplify problem-solving by specifying a numerical value of $|p(T)|$, for example.

2. Condition (d) is essential, because the function $p(t) \equiv 0$ obviously satisfies (a)–(c). Thus if we were to drop condition (d), every control \hat{u} would be an extremal: the definition would be utterly useless.

3. Condition (c) asserts that for each fixed t , the input value $\hat{u}(t)$ must maximize the product $p(t)^T Bv = B^T p(t) \bullet v$ over all v in U . Alternative notation for this is

$$\hat{u}(t) \in \arg \max \{H(x(t), p(t), v) : v \in U\}.$$

Geometrically, this means that the vector $B^T p(t)$ is an outward normal to U at the point $\hat{u}(t)$ —in symbols,

$$(c) \iff B^T p(t) \in N_U(\hat{u}(t)) \quad \text{a.e. } t \in [a, b].$$

[†] This abbreviation is commonly used in the theory of the Lebesgue integral, to stand for, “at all points outside some set of Lebesgue measure zero.” The theory being presented here extends without difficulty to the more advanced case, and the definition given here is appropriate with that understanding of “a.e.” as well.

All the trajectories of the control system that end up on the boundary of the attainable set can be found among its extremal pairs. We prove this next, along with some interesting extensions.

D.3. Theorem. *For any (control,state)-pair (\hat{u}, x) for system $(*)$ on $[0, T]$, the following are equivalent:*

- (i) $x(T) \in \text{bdy } \mathcal{A}(T; x(0), U)$.
- (ii) (\hat{u}, x) is extremal on $[0, T]$.
- (iii) (\hat{u}, x) is extremal on $[0, T]$, and the associated costate function p obeys

$$p(t) \in N_{\mathcal{A}(t; x(0), U)}(x(t)) \quad \forall t \in [0, T].$$

- (iv) $x(t) \in \text{bdy } \mathcal{A}(t; x(0), U)$ for each $t \in [0, T]$.

Proof. (i \Rightarrow ii) The set $\mathcal{A}(T; \xi, U)$ is convex by Prop. D.1. And $x(T) \notin \text{int } \mathcal{A}(T; \xi, U)$ by hypothesis. So by Thm. C.2(d), the set $\mathcal{A}(T; \xi, U)$ has at least one nonzero outward normal vector at $x(T)$. Pick one—call it $w \in N_{\mathcal{A}(T; x(0), U)}(x(T))$ —and use it to define $p(t) := e^{A^T(T-t)}w$. Since the matrix exponential is always invertible, the nontriviality of w guarantees that $p(t) \neq 0$ for all t . And, by direct calculation,

$$-\dot{p}(t) = A^T e^{A^T(T-t)} = A^T p(t) = \nabla_x H(x(t), p(t), \hat{u}(t)) \quad \text{a.e. } t \in [0, T].$$

Thus conditions (a), (b), and (d) in definition D.2 are all in force. Condition (c) is provided by Lemma D.1.

(ii \Rightarrow iii) Suppose (\hat{u}, \hat{x}) is extremal on $[0, T]$, and let p be an associated costate. Note that $p(t) \neq 0$ for all $t \in [0, T]$, by D.2(d). Rearrange the maximum condition, D.2(c):

$$\begin{aligned} p(t)^T B \hat{u}(t) &\geq p(t)^T B v && \forall v \in U \\ \iff 0 &\geq \langle p(t), B[v - \hat{u}(t)] \rangle && \forall v \in U \\ \iff p(t) &\in N_{\mathcal{A}(t; \hat{x}(0), U)}(\hat{x}(t)) && \text{(by Lemma D.1)}. \end{aligned}$$

(iii \Rightarrow iv) Apply Theorem C.2(d).

(iv \Rightarrow i) Obvious. ////

The previous result shows that *once a trajectory $x(\cdot)$ enters $\text{int } \mathcal{A}(t; \xi, U)$, no control can steer it back onto the boundary.* Trajectories which terminate on the boundary must evolve along the boundary for all time. This is why we call them “boundary trajectories.” Let’s look at some.

D.4. Example: The Rocket Car. Details in class.

Time-Optimal Control: Necessary Conditions. Typically the set U is bounded: in this case, $\mathcal{A}(t; \xi, U)$ is a *bounded* subset of the affine subspace

$$\mathcal{A}(t; \xi) = e^{At}\xi + \text{Im} [B \mid AB \mid \dots \mid A^{n-1}B].$$

It thus makes sense to consider the **minimum time problem**

$$\min \{T \geq 0 : \eta \in \mathcal{A}(T; \xi, U)\} \quad P(\xi, \eta)$$

Here the initial point ξ and the target point η are prescribed, and the goal is to drive the system from ξ to η as quickly as possible. We will focus on the most important target for stabilization, namely, $\eta = 0$.

D.5. Definition. For any subset S of \mathbb{R}^n , the *distance function* $x \mapsto \text{dist}(x; S)$ is defined by

$$\text{dist}(x; S) = \inf \{|x - s| : s \in S\}.$$

D.6. Theorem. Fix $\xi \in \mathbb{R}^n$ and let $U \subseteq \mathbb{R}^m$ be bounded. Then for each fixed $T > 0$, there exists a constant $K \geq 0$ such that

(a) for any $t_1, t_2 \in [0, T]$, one has

$$\text{dist}(y, \mathcal{A}(t_1; \xi, U)) \leq K|t_2 - t_1| \quad \forall y \in \mathcal{A}(t_2; \xi, U).$$

(b) for any $\eta \in \mathbb{R}^n$, the function $d(t) := \text{dist}(\eta; \mathcal{A}(t; \xi, U))$ obeys

$$|d(t_2) - d(t_1)| \leq K|t_2 - t_1| \quad \forall t_1, t_2 \in [0, T].$$

Proof. Choose $R > 0$ so large that the right side of the matrix norm inequality

$$\|e^{At} - e^{As}\| = \left\| e^{At} \left(I - e^{A(s-t)} \right) \right\| \leq e^{\|A\|T} \left(e^{\|A\||s-t|} - 1 \right)$$

is bounded above by $R|t - s|$ for any $s, t \in [0, T]$. Then choose $M > 0$ so large that

$$|e^{At}Bu| \leq M \quad \forall t \in [0, T], \forall u \in U.$$

We will use R and M to define K below.

(a) Fix $t_2 \in [0, T]$ and let $y \in \mathcal{A}(t_2; \xi, U)$. Then y is the endpoint $x(t_2)$ for some trajectory x with corresponding control u . Two cases arise: if $t_1 \leq t_2$, then the trajectory x provides a point $x(t_1)$ in $\mathcal{A}(t_1; \xi, U)$. Thus

$$\begin{aligned} & \text{dist}(y; \mathcal{A}(t_1; \xi, U)) \\ & \leq |x(t_2) - x(t_1)| \\ & = \left| \left(e^{At_2} - e^{At_1} \right) \xi + \int_{t_1}^{t_2} e^{A(t_2-s)} Bu(s) ds + \int_0^{t_1} \left(e^{A(t_2-t_1)} - I \right) e^{A(t_1-s)} Bu(s) ds \right| \\ & \leq R|\xi|(t_2 - t_1) + M(t_2 - t_1) + TMR(t_2 - t_1). \end{aligned}$$

If, on the other hand, $t_1 > t_2$, then choose any vector v in U and extend the control u generating x to the longer interval $[0, t_1]$ by setting $u(t) = v$ for all $t \in [t_2, t_1]$. In this case,

$$\begin{aligned} & \text{dist}(y; \mathcal{A}(t_1; \xi, U)) \\ & \leq \left| x(t_2) - \left(e^{At_1} \xi + \int_0^{t_2} e^{A(t_1-s)} Bu(s) ds + \int_{t_2}^{t_1} e^{A(t_1-s)} Bv ds \right) \right| \\ & = \left| (e^{At_2} - e^{At_1}) \xi + \int_0^{t_2} (I - e^{A(t_1-t_2)}) e^{A(t_2-s)} Bu(s) ds + \int_{t_2}^{t_1} e^{A(t_1-s)} Bv ds \right| \\ & \leq R|\xi|(t_1 - t_2) + TMR(t_1 - t_2) + M(t_1 - t_2). \end{aligned}$$

In both cases, we have the same estimate, and it leads to conclusion (a) with the constant $K = TMR + M + R|\xi|$.

(b) Fix $\eta \in \mathbb{R}^n$ and define the function d as indicated. Let $t_1, t_2 \in [0, T]$ be given. Then for any $\varepsilon > 0$, the set $\mathcal{A}(t_2; \xi, U)$ contains a point y satisfying $|y - \eta| \leq d(t_2) + \varepsilon$. By part (a), the set $\mathcal{A}(t_1; \xi, U)$ must contain a point x with $|x - y| \leq K|t_1 - t_2| + \varepsilon$. Therefore

$$d(t_1) \leq |\eta - x| \leq |\eta - y| + |y - x| \leq d(t_2) + \varepsilon + K|t_1 - t_2| + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$d(t_1) \leq d(t_2) + K|t_1 - t_2|.$$

Interchanging labels t_1 and t_2 and repeating the argument gives (b). ////

D.7. Theorem. Fix ξ in \mathbb{R}^n . Suppose $U \subseteq \mathbb{R}^m$ is compact and convex.

- (a) If $\eta \in \mathbb{R}^n$ and $T > 0$ obey $\eta \in \text{int } \mathcal{A}(T; \xi, U)$, then there exists $\delta > 0$ so small that $\eta \in \text{int } \mathcal{A}(t; \xi, U)$ for all t in $(T - \delta, T]$.
- (b) If the minimum-time problem $P(\xi, \eta)$ has a solution and T denotes the minimum time, then $\eta \in \text{bdy } \mathcal{A}(T; \xi, U)$. Therefore any minimizing control function must be extremal on $[0, T]$.

Proof. **(a)** (Contraposition.) If the conclusion were false, then there would be some sequence $T_k \rightarrow T^-$ such that

$$\eta \notin \text{int } \mathcal{A}(T_k; \xi, U) \quad \forall k.$$

Now each set $\mathcal{A}(T_k; \xi, U)$ is convex by Prop. C.4, so Thm. C.2 provides, for each k , a unit vector v_k satisfying

$$\langle v_k, \eta \rangle \geq \langle v_k, s \rangle \quad \forall s \in \mathcal{A}(T_k; \xi, U).$$

Now fix any y in $\mathcal{A}(T; \xi, U)$. By Thm. D.6(a), some point $s_k \in \mathcal{A}(T_k; \xi, U)$ satisfies

$$|s_k - y| \leq K|T_k - T| + \frac{1}{k}.$$

Thus

$$\langle v_k, \eta \rangle \geq \langle v_k, s_k \rangle = \langle v_k, s_k - y \rangle + \langle v_k, y \rangle \geq \langle v_k, y \rangle - K|T_k - T| - \frac{1}{k}.$$

Now the sequence v_k has a subsequence converging to some unit vector v . Taking the limit as $k \rightarrow \infty$ along this subsequence in the previous inequality, we find

$$\langle v, \eta \rangle \geq \langle v, y \rangle.$$

Since $y \in \mathcal{A}(T; \xi, U)$ is arbitrary, this inequality shows that $v \perp \mathcal{A}(T; \xi, U)$ at η . In particular, Thm. C.2(c) implies that $\eta \notin \text{int } \mathcal{A}(T; \xi, U)$. Contrapose.

(b) If $P(\xi, \eta)$ has a solution with minimum time T , then $\eta \in \mathcal{A}(T; \xi, U)$. If $\eta \in \text{int } \mathcal{A}(T; \xi, U)$, then statement (a) would contradict the minimality of T . Thus $\eta \in \text{bdy } \mathcal{A}(T; \xi, U)$. ////

Here is the main result of this section.

D.8. Theorem (Maximum Principle). *If the control set U is compact and convex, and the control function \hat{u} solves the minimum-time problem $P(\xi, \eta)$, then \hat{u} is extremal. That is, there exists a function p satisfying*

- (a) $-\dot{p}(t) = H_x(x(t), p(t), \hat{u}(t))^T = A^T p(t)$ a.e.,
- (b) $\dot{x}(t) = H_p(x(t), p(t), \hat{u}(t))^T = Ax(t) + B\hat{u}(t)$ a.e.,
- (c) $H(x(t), p(t), \hat{u}(t)) \geq H(x(t), p(t), v)$ for all $v \in U$, a.e.,
- (d) $p(T) \neq 0$.

Modified Problems and Transversality Conditions. The theory developed here can be applied to other problems as well. Here are two related situations. In both cases, we assume that U is compact and convex.

A. Minimum time to reach a given set S . In standard form, this problem reads

$$\begin{aligned} \min \{ T > 0 : \dot{x} = Ax + Bu, x(0) = \xi, \\ u(t) \in U \text{ a.e.}, \\ x(T) \in S \}. \end{aligned} \quad P(\xi, S)$$

It can be reduced to a problem involving the motion of the attainable sets in \mathbb{R}^n as follows:

$$\min \{ T > 0 : \mathcal{A}(T; \xi, U) \cap S \neq \emptyset \}.$$

If this minimum is attained, and has the value T , there must be some point η in $S \cap \mathcal{A}(T; \xi, U)$. Of course, $\eta = x(T)$ for some state process x associated with the optimal control \hat{u} . Now since $\eta \in S$, \hat{u} most certainly solves the ordinary minimum-time problem $P(\xi, \eta)$. (If there were any faster way to steer the system from ξ to η , it

would also be a faster way to hit the target set S .) Therefore $\eta \in \text{bdy } \mathcal{A}(T; \xi, U)$ by Thm. D.7, and \hat{u} must be an extremal in the sense of Def. C.5. It is also clear from the arguments underlying Thm. D.7 that η cannot be an interior point of the set S . (Why?) When S is convex, the fact that $\eta \in (\text{bdy } S) \cap (\text{bdy } \mathcal{A}(T; \xi, U))$ implies (via a small extension of Thm. C.2(c)) that there exists a unit vector $w \in \mathbb{R}^n$ satisfying

$$\text{both } w \perp \mathcal{A}(T; \xi, U) \quad \text{and} \quad -w \perp S \text{ at } \eta = x(T).$$

This allows us to say more about the adjoint function p we choose to describe the extremality of the optimizing control \hat{u} . In view of Thm. D.3(a), we can replace the nontriviality condition (d) in Def. C.5 with a more informative assertion about $p(T)$. This statement, D.9(d) below, is called a **transversality condition**.

D.9. Theorem (Maximum Principle for $P(\xi, S)$). *If the control set U is compact and convex, and the control function \hat{u} solves the minimum-time problem $P(\xi, S)$ for a convex set S , then there exists a continuous function p satisfying*

- (a) $-\dot{p}(t) = H_x(x(t), p(t), \hat{u}(t))^T = A^T p(t)$ a.e.,
- (b) $\dot{x}(t) = H_p(x(t), p(t), \hat{u}(t))^T = Ax(t) + B\hat{u}(t)$ a.e.,
- (c) $H(x(t), p(t), \hat{u}(t)) \geq H(x(t), p(t), v)$ for all $v \in U$, a.e.,
- (d) $p(T) \neq 0$ and $-p(T) \perp S$ at $x(T)$.

(In simple problems, the vectors perpendicular to S are easy to find.)

B. A fixed-time problem. Here the terminal time $T > 0$ is given, along with a smooth function $\ell: \mathbb{R}^n \rightarrow \mathbb{R}$. The problem is

$$\min \{ \ell(x(T)) : \dot{x} = Ax + Bu, x(0) = \xi, \\ u(t) \in U \text{ a.e.} \}.$$

It can be reduced to a problem involving attainable sets as follows:

$$\min \{ \ell(\eta) : \eta \in \mathcal{A}(T; \xi, U) \}.$$

D.10. Lemma. *Let \mathcal{A} be any convex subset of \mathbb{R}^n . If $\ell: \mathbb{R}^n \rightarrow \mathbb{R}$ is smooth and*

$$\hat{\eta} \in \arg \min \{ \ell(\eta) : \eta \in \mathcal{A} \},$$

then either $\nabla \ell(\hat{\eta}) = 0$ or else $-\nabla \ell(\hat{\eta}) \perp \mathcal{A}$ at $\hat{\eta}$.

Proof. Suppose $\hat{\eta} \in \mathcal{A}$ is a point where $\nabla \ell(\hat{\eta})$ is not zero, and is not perpendicular to \mathcal{A} at $\hat{\eta}$. This means that there is some point $a \in \mathcal{A}$ for which

$$\begin{aligned} \langle -\nabla \ell(\hat{\eta}), a \rangle &> \langle -\nabla \ell(\hat{\eta}), \hat{\eta} \rangle \\ &\iff \langle \nabla \ell(\hat{\eta}), a - \hat{\eta} \rangle < 0 \\ &\iff \lim_{t \rightarrow 0^+} \frac{\ell(\hat{\eta} + t(a - \hat{\eta})) - \ell(\hat{\eta})}{t} \\ &\implies \ell(\hat{\eta} + t(a - \hat{\eta})) < \ell(\hat{\eta}) \text{ for all } t > 0 \text{ sufficiently small} \\ &\implies \hat{\eta} \notin \arg \min \{ \ell(\eta) : \eta \in \mathcal{A} \}. \end{aligned}$$

(The last implication relies upon the convexity of \mathcal{A} to ensure that $\widehat{\eta} + t(a - \widehat{\eta}) \in \mathcal{A}$ for all $t \in (0, 1)$.) ////

This observation lets us write down a version of the maximum principle for the fixed-time problem.

D.11. Theorem (Another Maximum Principle). *Let U be a compact convex set, and let \widehat{u} solve the fixed-time problem above. Then there exists a function p such that*

- (a) $-\dot{p}(t) = H_x(x(t), p(t), \widehat{u}(t))^T = A^T p(t)$ a.e.,
- (b) $\dot{x}(t) = H_p(x(t), p(t), \widehat{u}(t))^T = Ax(t) + B\widehat{u}(t)$ a.e.,
- (c) $H(x(t), p(t), \widehat{u}(t)) \geq H(x(t), p(t), v)$ for all $v \in U$, a.e.,
- (d) $-p(T) = \nabla \ell(x(T))$.

Proof. Notice that the choice $p(\cdot) \equiv 0$ satisfies Def C.5(a)–(c) and gives the first possible conclusion of the lemma above. If the second conclusion of the lemma above holds, then the choice $p(t) = e^{A^T(T-t)} \nabla \ell(x(T))$ satisfies (d) by construction and obeys C.5(a)–(c) by virtue of Thm. D.3(a). ////

Time-Optimal Control: Sufficient Conditions. Although Theorem D.3 gives a complete characterization of boundary trajectories, its counterpart in the theory of necessary conditions (Thm. D.8) does not completely characterize the time-optimal arcs. We know that every optimal control must satisfy conditions D.8(a)–(d), but we do not know if these conditions are also satisfied by some non-optimal controls (spurious “solutions”). This is because neither theorem address the possibility that some fixed point of \mathbb{R}^n might remain on the boundary of $\mathcal{A}(t; \xi, U)$ throughout an interval of time. We account for this possibility in Cor. D.13 below. That result depends upon the following “controllability result”, which provides a qualitative analogue of Thm. A.1(a).

D.12. Proposition. *For any control set $U \subseteq \mathbb{R}^m$, if $0 \in \text{int } U$ and (A, B) is controllable, then*

$$0 \in \text{int } \mathcal{A}(t; 0, U) \text{ for all } t > 0.$$

Proof. By assumption, U contains the closed ball $\mathbb{B}[0; \varepsilon]$ for some $\varepsilon > 0$. Let $\widetilde{U} = \mathbb{B}[0; \varepsilon]$: then $\widetilde{U} \subseteq U$, so $\mathcal{A}(t; 0, \widetilde{U}) \subseteq \mathcal{A}(t; 0, U)$. It suffices to prove that $0 \in \text{int } \mathcal{A}(t; 0, \widetilde{U})$ for all $t > 0$.

Suppose not. If $0 \notin \text{int } \mathcal{A}(T; 0, \widetilde{U})$ for some $T > 0$, then by Thm. C.2(c) there

must be some $w \neq 0$ such that

$$\begin{aligned} \langle w, 0 \rangle &\geq \langle w, y(T) \rangle \quad \forall y(T) \in \mathcal{A}(T; 0, U) \\ \iff 0 &\geq w^T e^{AT} \int_0^T e^{-At} B u(t) dt \quad \forall u \in PWC([0, T]; \tilde{U}). \end{aligned}$$

Define $v^T = w^T e^{AT}$. Note that the set \tilde{U} is symmetric about the origin, so the previous inequality actually implies

$$0 = v^T \int_0^T e^{-At} B u(t) dt \quad \forall u \in PWC([0, T]; \tilde{U}).$$

Since the right side is linear in $u(\cdot)$, we can scale it to deduce that

$$0 = v^T \int_0^T e^{-At} B u(t) dt \quad \forall u \in PWC([0, T]; \mathbb{R}^m).$$

Just as in the proof of Thm. 1.1, this implies

$$v^T [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B] = 0^T,$$

which contradicts the controllability of (A, B) . This is impossible, so the theorem must hold. ////

A sufficient condition for the minimum-time problem $P(\xi, 0)$ follows directly. (The case of $P(\xi, \eta)$ with $\eta \neq 0$ is somewhat harder to treat, and we omit it.)

D.13. Corollary. *Let $U \subseteq \mathbb{R}^m$ be compact and convex. Suppose also that $0 \in \text{int } U$ and that (A, B) is controllable. Then for any $T > 0$ and $\hat{u} \in PWC([0, T]; U)$ for which the response satisfies $x(T) = 0$, the following assertions are equivalent:*

- (a) \hat{u} is extremal on $[0, T]$;
- (b) \hat{u} solves problem $P(\xi, 0)$.

Proof. (a \Rightarrow b) We prove that “not (b)” implies “not (a)”. Thus, we assume that $x(T) = 0$, but that \hat{u} is not optimal. This means that there is some time $T_0 < T$ for which $0 \in \mathcal{A}(T_0; \xi, U)$. This implies that for any time $t > T_0$, $\mathcal{A}(t; \xi, U) \supseteq \mathcal{A}(t - T_0; 0, U)$. (Why?) We apply this observation with $t = T$ to get

$$0 \in \text{int } \mathcal{A}(T - T_0; 0, U) \subseteq \text{int } \mathcal{A}(T; \xi, U).$$

(The first inclusion follows from Prop. D.12.) This implies that 0 is not a boundary point of $\mathcal{A}(T; \xi, U)$, so we know that the control \hat{u} cannot be extremal on $[0, T]$ by Thm. D.3.

(b \Rightarrow a) Proved above as Cor. D.5. ////

A Survey of Advanced Topics

Existence Theory. We would like to have a theorem something like this:

D.14. Desirable Statement. *Let $U \subseteq \mathbb{R}^m$ be compact convex. Consider the minimum time problem $P(\xi, \eta)$. If $\eta \in \mathcal{A}(T; \xi, U)$ for some $T > 0$, then $P(\xi, \eta)$ has a solution.*

If we could show that $\mathcal{A}(t; \xi, U)$ is closed for each $t > 0$, then D.14 would be easy to prove. We would simply consider the function

$$d(t) := \text{dist}(\eta; \mathcal{A}(t; \xi, U))$$

first mentioned in Thm. C.5(b). That result showed that d is continuous on $[0, +\infty)$. Hence the set $\{t \geq 0 : d(t) = 0\}$ is a closed set, which contains some $T > 0$ by assumption. Hence this set contains a smallest element, say \hat{T} . Then for $t < \hat{T}$ we have $d(t) > 0$, so $\eta \notin \mathcal{A}(t; \xi, U)$, while for \hat{T} we have $0 = d(\hat{T}) = \text{dist}(\eta; \mathcal{A}(\hat{T}; \xi, U))$. If $\mathcal{A}(\hat{T}; \xi, U)$ is closed, then this implies $\eta \in \mathcal{A}(\hat{T}; \xi, U)$ and D.14 is proved.

Unfortunately, our assumptions do not imply that $\mathcal{A}(t; \xi, U)$ is always a closed set. An example given by D. B. Silin, Soviet Math. Doklady **23**(1981), 309–311 has $m = n = 3$, $B = I$, and $U \subseteq \mathbb{R}^3$ compact convex, but $\mathcal{A}(T; \xi, U)$ is not closed for certain times $T > 0$. Silin modifies his example to formulate a version of $P(\xi, 0)$ which satisfies the hypotheses of D.14, but has no solution. Some additional hypotheses are clearly required. Here are two workable alternatives.

Approach 1 (Specialize the control set). Silin (1981, op. cit.) states that if U is compact, convex, and polyhedral, then the sets $\mathcal{A}(t; \xi, U)$ are closed for $t \geq 0$. (“Polyhedral” means that U is the intersection of finitely many affine half-spaces, i.e., $U = \{v \in \mathbb{R}^m : \langle g_i, v \rangle \leq c_i\}$ for a finite collection of vectors $g_1, \dots, g_N \in \mathbb{R}^m$ and constants c_1, \dots, c_N .) This covers most applications, and turns D.14 into a theorem.

Approach 2 (Generalize the control functions). If we consider Lebesgue measurable control functions $u(\cdot)$ instead of merely the piecewise continuous control functions, we obtain a larger attainable set:

$$\mathcal{A}_m(t; \xi, U) = \left\{ e^{At}\xi + \int_0^t e^{A(t-s)}Bu(s) ds : u \in L^\infty([0, t]; U) \right\}.$$

It can be shown that for a compact convex set U ,

$$\mathcal{A}_m(t; \xi, U) = \text{cl } \mathcal{A}(t; \xi, U).$$

Hence the sets $\mathcal{A}_m(t; \xi, U)$ are closed, and not too much larger than the sets we have used. Our necessary and sufficient conditions for optimality all go through in this setting (except that the meaning of “a.e.” changes from “from all but finitely many points” to “almost everywhere in the Lebesgue sense”), and statement D.14 again becomes a valid theorem.

Appendix: Bang-Bang Controls. In all of our examples, the optimal controls we found took on values on the boundary of the control set U . Is it possible to use a smaller control set and obtain the same attainable sets? A positive answer would be useful to anyone who has to build an optimal control law, since it would be necessary to implement only a subset of the possible control values.

D.15. Definition. Given a set $S \subseteq \mathbb{R}^m$, the *convex hull* of S is the smallest convex set containing S , denoted $\text{co } S$. Analytically,

$$\text{co } S = \left\{ \sum_{i=1}^r \lambda_i s_i : r \in \mathbf{N}, \lambda_i \geq 0, \sum_{i=1}^r \lambda_i = 1, s_i \in S \right\}.$$

A *multifunction* (short for “multi-valued function”) is a mapping whose values are not points, but sets. One example is the mapping $t \mapsto \mathcal{A}(t; \xi, U)$ studied earlier. Another is the multifunction

$$\Gamma(t) = e^{-At}BU := \{e^{-At}Bu : u \in U\}.$$

The *integral of a multifunction* $\Gamma: \mathbb{R} \rightrightarrows \mathbb{R}^n$ is defined as follows:

$$\int_a^b \Gamma(t) dt = \left\{ \int_a^b \gamma(t) dt : \gamma: [a, b] \rightarrow \mathbb{R}^n \text{ is integrable, and } \gamma(t) \in \Gamma(t) \text{ a.e. } t \in [a, b] \right\}.$$

A multifunction $\Gamma: \mathbb{R} \rightrightarrows \mathbb{R}^n$ is *measurable* if the set below is measurable for any open set $\Omega \subseteq \mathbb{R}^n$:

$$\{t \in \mathbb{R} : \Gamma(t) \cap \Omega \neq \emptyset\}.$$

D.16. Theorem (Aumann). *If $\Gamma: [a, b] \rightrightarrows \mathbb{R}^n$ is measurable, and its values are nonempty compact sets, and there exists $R > 0$ so big that*

$$\Gamma(t) \subseteq \mathbb{B}[0; R] \text{ a.e. } t \in [a, b],$$

then one has

$$\int_a^b \Gamma(t) dt = \int_a^b \text{co } \Gamma(t) dt,$$

and both these sets are nonempty, compact, and convex.

Our interest in Aumann’s theorem derives from the following observation. Suppose that U is compact and convex, while V is a compact set for which $U = \text{co } V$.

Then for any $T > 0$,

$$\begin{aligned}
\mathcal{A}_m(T; \xi, V) &= e^{AT} \xi + e^{AT} \left\{ \int_0^T e^{-At} B u(t) dt : u \in L^\infty([0, T]; V) \right\} \\
&= e^{AT} \xi + e^{AT} \int_0^T \Gamma(t) dt, \text{ where } \Gamma(t) := e^{-At} B V, \\
&= e^{AT} \xi + e^{AT} \int_0^T \text{co } \Gamma(t) dt, \text{ by Aumann's theorem,} \\
&= e^{AT} \xi + e^{AT} \left\{ \int_0^T e^{-At} B u(t) dt : u \in L^\infty([0, T]; U) \right\} \\
&= \mathcal{A}_m(T; \xi, U).
\end{aligned}$$

Thus any point reachable using measurable controls in U can be reached in the same time using measurable controls in the smaller set V .

When V is the smallest closed subset of U such that $U = \text{co } V$, then the functions in $L^\infty([0, T]; V)$ are called *bang-bang controls*.

To identify the set of bang-bang controls, note first that any set V such that $U = \text{co } V$ must contain all the “extreme points” of U : these are the vectors \hat{u} which have the following property:

$$\text{If } \hat{u} = \frac{u_1 + u_2}{2} \text{ for some } u_1, u_2 \in U, \text{ then } u_1 = u_2 = \hat{u}.$$

(In other words, \hat{u} is an extreme point of U iff it cannot be realized as the midpoint of a nontrivial line segment contained in U .) The extreme points of U are denoted $\text{ext } U$. The famous Krein-Milman theorem asserts that if U is compact and convex, then $U = \text{cl co ext } U$. It follows immediately that the smallest closed set V with the property that $\text{co } V = U$ is the set $V = \text{cl ext } U$. Thus we have the following result.

D.17. Theorem (The Bang-Bang Theorem). *If U is compact and convex, then for every $T > 0$, one has*

$$\mathcal{A}_m(T; \xi, U) = \mathcal{A}_m(T; \xi, \text{cl ext } U).$$

In particular, if the min-time problem $P(\xi, \eta)$ with measurable controls has a solution, then it has a bang-bang solution—i.e., a solution $\hat{u} \in L^\infty([0, T]; \text{cl ext } U)$.

Notice that Theorem D.17 applies only to individual optimal control functions $\hat{u}(t)$, and not to nonlinear feedback laws of the type we seek in typical problems. Example 17.4 in Hermes and LaSalle, *Functional Analysis and Time-Optimal Control*, illustrates this point by presenting a problem with a solution that is not bang-bang. We note, however, that the solution in this problem is not unique, and that there is still a feedback solution of bang-bang type. This raises the interesting research question: Can the bang-bang theorem be extended to assert that there exists a function $\hat{u}: \mathbb{R}^n \rightarrow \text{cl ext } U$ such that the discontinuous feedback control dynamics $\dot{x} = Ax + B\hat{u}(x)$ have a solution for every initial point ξ in \mathbb{R}^n , and the solution is always precisely the time-optimal trajectory from ξ to 0?