Handwritten notes are allowed. Write your answers in the booklet provided. Throughout this paper, the Basic Problem is

$$\min_{x \in PWS} \left\{ \Lambda[x] := \int_a^b L(t, x(t), \dot{x}(t)) \, dt : x(a) = A, \, x(b) = B \right\}.$$ \hfill (P)

We assume throughout that $L \in C^2$.

[20] 1. Among all piecewise smooth functions $x: [0, 1] \to \mathbb{R}$ satisfying $x(0) = 0$, find the unique global minimizer of

$$\Lambda[x] = \int_0^1 (\dot{x}(t)^2 + 4x(t)^2 + 8x(t)) \, dt.$$  

There is no constraint on $x(1)$. Explain why the arc you find has the stated minimality properties.

[20] 2. Let $L(t, x, v) = f(x)^2 v^2$ for some positive-valued function $f(x)$ of class $C^2$. Consider the basic problem on the interval $[a, b]$.

(a) Prove that every extremal is of class $C^2$.

(b) Prove that every extremal must be either strictly increasing, strictly decreasing, or constant, in the entire interval $[a, b]$.

(c) Find the unique admissible extremal in the specific case where $L(x, v) = e^{2x} v^2$, $(a, A) = (1, 0)$, $(b, B) = (2, \ln 4)$.

(d) Either show that the arc from part (c) is a global minimizer, or outline one reasonable approach to proving minimality and show where it breaks down.

[20] 3. For $\Lambda[x]$ as in the basic problem, and $\Gamma[x] := \int_a^b G(t, x(t), \dot{x}(t)) \, dt$, consider the problem

$$\min_{x \in PWS} \left\{ \Lambda[x] : x(a) = A, \, x(b) = B, \, \Gamma[x] \leq C \right\}.$$ \hfill (Q)

Here $A$, $B$, and $C$ are given constants, and we assume $G \in C^2$.

(a) Suppose $\hat{x}$ satisfies the constraints in $(Q)$, with $\Gamma[\hat{x}] = C$. Prove: If there exists some $\lambda \geq 0$ such that $\hat{x}$ provides a minimum in the version of the Basic Problem $(P)$ where $\tilde{L} = L + \lambda G$, then $\hat{x}$ provides a minimum in problem $(Q)$.

(b) Find an absolute minimizer and discuss uniqueness:

$$\min \left\{ \int_0^1 \frac{1}{2} \dot{x}(t)^2 \, dt : x(0) = 0, \, x(1) = 6, \ \int_0^1 12t^2 x(t) \, dt \leq \frac{90}{7} \right\}.$$