1. Carefully sketch the nine lines in the first quadrant joining \((0, \alpha)\) to \((1 - \alpha, 0)\), where \(\alpha = 0.1, 0.2, \ldots, 0.9\). The picture suggests that the family of all such lines, with \(\alpha\) taking values in \((0, 1)\), forms an envelope: substantiate this by finding the envelope’s equation.

2. For fixed \(b > 0\) and \(B \in \mathbb{R}\), consider the basic problem

\[
\min \left\{ \int_0^b \left( \dot{x}^4(t) - 6\dot{x}^2(t) - 2x(t)\dot{x}(t) \right) \, dt : x(0) = 0, \ x(b) = B \right\}
\]

(a) Consider the set of admissible extremals in \(\hat{C}^1\). Show that its intersection with \(C^1\) is a single arc.

(b) Show that the smooth extremal singled out in (a) gives a global minimum if \(|B|/b \geq \sqrt{3}\), but not even a strong local minimum if \(|B|/b < \sqrt{3}\).

(c) Assuming \(|B|/b < \sqrt{3}\), describe as completely as possible the set of all admissible extremals having one or more corner points. Show that every such curve gives a global minimum.

(d) Give a formula for the **Value Function** \(V\), defined on \((0, \infty) \times \mathbb{R}\) by

\[
V(b, B) = \text{minimum value above, with endpoint } (b, B).
\]

Hint: Simplify the integral; then use your calculus skills on \(f(v) \overset{\text{def}}{=} v^4 - 6v^2\).

3. In each situation below, find all admissible extremals. Classify each one as a weak local optimum, a strong local optimum, or a global optimum, replacing “optimum” by “maximum” or “minimum” as appropriate. Justify your classification; make the strongest assertion you can.

(a) \[ \int_0^{\pi/4} \left[ 4x(t)^2 - \dot{x}(t)^2 + 8x(t) \right] \, dt \text{ subject to } x(0) = -1, \ x(\pi/4) = 0, \]

(b) \[ \int_0^1 \left[ \dot{x}(t)^2 + x(t)^2 + 2e^{2t}x(t) \right] \, dt \text{ subject to } x(0) = 1/3, \ x(1) = e^2/3. \]

(c) \[ \int_{-1}^2 \dot{x}(t) \left( 1 + t^2 \dot{x}(t) \right) \, dt \text{ subject to } x(-1) = 1, \ x(2) = 1. \]

4. Consider \(\Lambda[x] := \int_0^1 \left[ \dot{x}(t)^2 - 4x(t)\dot{x}(t)^3 + 2t\dot{x}(t)^4 \right] \, dt \text{ subject to } x(0) = 0, \ x(1) = 0\). Prove that \(\dot{x} \equiv 0\) is a weak local minimum and satisfies \(\mathcal{E}(t, \dot{x}(t), \ddot{x}(t), w) \geq 0, \) but that \(\dot{x}\) is not a strong local minimum. (Thus a weak local minimum satisfying (W) need not be a strong local minimum. Note: It’s easy to show that (W*) fails along \(\dot{x}\), but this is not decisive ... it only shows that one possible method for establishing strong local minimality fails to apply. A direct argument is required.)

(continued ... )
5. (Troutman 7.16.) Let $L(x, v) = 3x^2 + v^4$.

(a) Show that if $x \in C^1[-2, 2]$, then (WE2) implies $(\dot{x}^2 - x) (\dot{x}^2 + x) = c$ for some constant $c$.

(b) Take $c = 0$ in (a), and find a $C^1$ arc satisfying (WE2) together with $x(-2) = -1, x(2) = 1$.
   [Hint: Patch together solutions which make a factor in (a) vanish.]

(c) Show that the function $L$ is strictly convex. Deduce that the arc $\tilde{x}$ found in (b) gives the unique global minimizer in the basic problem with integrand $L$ and endpoint conditions from (b).

(d) Show that $\tilde{x}$ is not $C^2$. Explain why this does not contradict Hilbert’s Regularity Theorem.