1. Given a function $N$ continuous and positive-valued on $[0, 1]$, find the integrable function $u : [0, 1] \to \mathbb{R}$ that gives the minimum value to the integral

$$\int_0^1 N(t)u(t)^2 \, dt,$$

subject to the controlled differential equation constraint

$$\dot{x}(t) = -x(t) + u(t), \quad x(0) = A, \; x(1) = B.$$

Include simple integral formulas for any constants that may appear; explain why the function you specify produces a true minimum.

2. (a) Prove: if $G \in C^1$ and $x \in C^2$ satisfy $\dot{x}(t)^2 = G(x(t))$ for all $t$ in $(a, b)$, and $\dot{x}(t)$ has only finitely many zeros, then

$$\ddot{x}(t) = \frac{1}{2}G'(x(t)) \quad \forall \; t \in (a, b).$$

(b) Consider the Lagrangian $L(x, v) = f(x)\sqrt{1 + v^2}$, where $f \in C^1$. Let $x$ be an extremal arc for $L$ on $(a, b)$, with the property that $f(x(t)) > 0$ for all $t \in (a, b)$. Deduce that there is some $c > 0$ such that

$$\ddot{x}(t) = c^2 f(x(t))f'(x(t)) \quad \forall \; t \in (a, b).$$

(c) Now specialize to the Lagrangian $L(x, v) = \sqrt{(\alpha x^2 + 2\beta x + \gamma)(1 + v^2)}$. Find the general form of the extremals in regions where the radicand is positive. [Fine point: The first steps of this approach produce families of curves involving the (unknown) constant $c$ from part (b) and two arbitrary constants $A$ and $B$. Evaluate $c$ in terms of $A$ and $B$.] For the three special cases $(\alpha, \beta, \gamma) = (1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, briefly describe the context in which you have met the problem and its extremals before, including geometrical interpretations where possible.

3. Solve as completely as possible, making appropriate use of our refined convexity-based sufficiency theory:

$$\min \left\{ \Lambda[x] := \int_0^1 -x(t)\sqrt{1 - \dot{x}(t)^2} \, dt : x(0) = 0, \; x(t) > 0 \; \forall t \in (0, 1) \right\}.$$

Notice that the right endpoint $x(1)$ is unconstrained.
4. (a) Formulate a variational problem in which a minimizing arc $x$ would automatically solve the following boundary-value problem:

$$\ddot{x}(t) - r(t)x(t)^3 + p(t) = 0, \quad 0 < t < 1; \quad x(0) = A, \quad \dot{x}(1) = 0. \quad (\ast)$$

You may assume that $r$ and $p$ are continuously differentiable on $[0, 1]$.

(b) Prove that if $r(t) \geq 0$ for $0 \leq t \leq 1$, then any solution of $(\ast)$ will give a true minimizer in the variational problem found in (a).

Remark: Reformulating a given boundary-value problem as a minimization problem in function space is a standard method in the numerical solution of both ordinary and partial differential equations.

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**REMARKER:** Midterm test in class on Wednesday 25 February 2015

Students may bring any papers or notes written in their own hand. Books or printed materials are not allowed. Calculators are banned.