1. An airplane must fly between two cities separated by a distance \( D \). The cost of flying a distance (arc length) \( ds \) at an altitude \( h \) equals \( \exp(-h/H) \) \( ds \), where \( H > 0 \) is a given constant. Find the only possible candidate for the cheapest flight path. (Make the flat-earth approximation.)

BONUS: Taking \( H = 1 \), describe and sketch the infimum cost as a function of \( D > 0 \). (Assume your candidate gives the minimum cost.)

2. Find the form of the vector-valued extremals for these variational integrals:
   
   (a) \( \int [\dot{x}(t)^2 + \dot{x}(t)\dot{y}(t) + \dot{y}(t)^2] \, dt \).
   
   (b) \( \int [\dot{x}(t)^2 - \dot{y}(t)^2 + 2x(t)y(t) - 2x(t)^2] \, dt \).

3. [Tricksy.] Solve the following problem, show that the answer is not unique, and discuss:

   Find a cubic polynomial that provides a global minimum for the functional
   
   \[ \Lambda[x] := \int_0^1 \left( \frac{9}{4} t^2 x(t)^4 + 3t^3 x(t)^3 \dot{x}(t) \right) \, dt \]
   
   subject to the endpoint requirements \( x(0) = 0 = x(1) \).

4. Suppose (only) that \( P: [a, b] \to \mathbb{R} \) is continuous. Prove that the following are equivalent (TFAE):

   (a) \( \int_a^b P(t)\ddot{h}(t) \, dt = 0 \) for each “variation” \( h \in C^2([a, b]) \) satisfying

   both \( h(a) = 0 = h(b) \) and \( \dot{h}(a) = 0 = \dot{h}(b) \).

   (b) \( P(t) = mt + c \) for some constants \( m \) and \( c \).

   Caution: The set of all arcs \( k \overset{\text{def}}{=} \ddot{h} \) generated by variations as described in (a) is a proper subspace of \( V_{II} \). (Reason: Every such \( k \) obeys \( \int_a^b k(r) \, dr = 0 \), but many elements of \( V_{II} \) do not.) Therefore this problem cannot be solved simply by substituting \( k = \ddot{h} \) and citing the lemma of DuBois-Reymond. A more direct approach, extending the proof of DuBois-Reymond’s Lemma, will be needed.

5. Let \( X = C^2[a, b] \) and consider the functional \( M: X \to \mathbb{R} \) defined by

   \[ M[x] := \int_a^b L(t, x(t), \dot{x}(t), \ddot{x}(t)) \, dt, \quad \text{for all } x \in X. \]
Here the (given) integrand $L(t, x, v, w): [a, b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of class $C^1$.

(a) Suppose that for some $\hat{x} \in X$, we know that the directional derivative $M'[\hat{x}; h] = 0$ for every $h$ in the subspace of $X$ defined by the four conditions $h(a) = 0 = h(b)$ and $\dot{h}(a) = 0 = \dot{h}(b)$. By introducing such functions as

$$\phi(t) = \int_a^t \hat{L}_v(r) \, dr, \quad \mu(t) = \int_a^t \hat{L}_x(r) \, dr,$$

find an integro-differential equation satisfied by $\hat{x}$. Check that it reduces to (IEL) when $L$ is independent of $w$.

(b) If both $L$ and $\hat{x}$ were known to be sufficiently smooth, repeated differentiation would reduce the equation in (b) to a fourth-order ordinary differential equation for $\hat{x}$. Find this equation. Check that it reduces to (DEL) when $L$ is independent of $w$.

(c) Among all curves $x$ in $X$ that join $\alpha = (0, 0)$ to $\beta = (1, 0)$ and satisfy $\dot{x}(0) = 1$ and $\dot{x}(1) = -1$, find the one that minimizes

$$M[x] := \int_0^1 (\dddot{x}(t))^2 \, dt.$$

Be sure to prove that your candidate really gives the minimum. (Hint: Use the ODE in (b) to find a candidate, then proceed directly.)

Hint: Question 4 will help.