[12] **1.** Consider the linear system below. There are no nonnegativity requirements.

$$\begin{aligned} 8x_1 + 2x_2 - x_3 - x_4 - 3x_5 &= 3 \\ -5x_1 - x_2 + x_3 + 3x_4 + 2x_5 &= 4 \\ 10x_1 \phantom{{}+2x_2} - x_3 - 3x_4 - 4x_5 &= 0 \end{aligned} \qquad (*)$$

Solve parts (a)–(c) below. You may find the following identity useful:

$$B = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix} \iff B^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}.$$

(a) Write a dictionary representation of the system $(*)$ with respect to the basis of your choice.

(b) Determine the values of x_1, x_2, and x_5 for which one solution of $(*)$ is $\mathbf{x} = (x_1, x_2, 2, -2, x_5)$.

(c) Find, in terms of r and s, the values of x_1, x_2, and x_5 for which one solution of $(*)$ is $\mathbf{x} = (x_1, x_2, 2r, 2s, x_5)$.

(a) The given matrix B resembles the coeffs of x_1, x_3, x_5. So write

$$(*) \Leftrightarrow \begin{cases} 8x_1 - x_3 - 3x_5 = 3 - 2x_2 + x_4 \\ -5x_1 + x_3 + 2x_5 = 4 + x_2 - 3x_4 \\ 10x_1 - x_3 - 4x_5 = 0 \phantom{{}+x_2} + 3x_4 . \end{cases}$$

Recognize LHS as $B\begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix}$, so apply B^{-1} to both sides:

$$\begin{aligned} x_1 &= 10 - 3x_2 - 4x_4 \\ x_3 &= 8 + 2x_2 - 3x_4 \\ x_5 &= 23 - 8x_2 - 10x_4 . \end{aligned}$$

That's a dictionary for (*) in which x_1, x_3, x_5 are the basic vars.

(b) Put $x_3 = 2$, $x_4 = -2$ into the dict from (a) to get

$$\begin{aligned} x_1 &= 18 - 3x_2 && \cdots\cdots\to \quad x_1 = +36 \\ 2 &= 14 + 2x_2 && \Rightarrow x_2 = -6 \\ x_5 &= 43 - 8x_2 && \cdots\cdots\to \quad x_5 = 91 \end{aligned}$$

One solution for (*) is $\vec{x} = (+36, -6, 2, -2; 91)$.

Continued on page 3

[12] **1.** Consider the linear system below. There are no nonnegativity requirements.

$$\begin{aligned} 8x_1 + 2x_2 - x_3 - x_4 - 3x_5 &= 3 \\ -5x_1 - x_2 + x_3 + 3x_4 + 2x_5 &= 4 \\ 10x_1 \qquad - x_3 - 3x_4 - 4x_5 &= 0 \end{aligned} \qquad (*)$$

Solve parts (a)–(c) below. You may find the following identity useful:

$$B = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix} \iff B^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}.$$

(a) Write a dictionary representation of the system $(*)$ with respect to the basis of your choice.

(b) Determine the values of x_1, x_2, and x_5 for which one solution of $(*)$ is $\mathbf{x} = (x_1, x_2, 2, -2, x_5)$.

(c) Find, in terms of r and s, the values of x_1, x_2, and x_5 for which one solution of $(*)$ is $\mathbf{x} = (x_1, x_2, 2r, 2s, x_5)$.

Continued ...

(c) Pivot the dict in (a) to make x_2 enter basis, x_3 leave:

$$-2x_2 = 8 - x_3 - 3x_4 \iff x_2 = -4 + \tfrac{1}{2}x_3 + \tfrac{3}{2}x_4$$

$$x_1 = 10 - 3x_2 - 4x_4 \quad +12 - \tfrac{3}{2}x_3 - \tfrac{9}{2}x_4 \iff x_1 = 22 - \tfrac{3}{2}x_3 - \tfrac{17}{2}x_4$$

$$x_5 = 23 - 8x_2 - 10x_4 \quad +32 - 4x_3 - 12x_4 \iff x_5 = 55 - 4x_3 - 22x_4$$

Now use $x_3 = 2r$, $x_4 = 2s$:

$$x_1 = 22 - \tfrac{3}{2}(2r) - \tfrac{17}{2}(2s) = 22 - 3r - 17s$$

$$x_2 = -4 + \tfrac{1}{2}(2r) + \tfrac{3}{2}(2s) = -4 + r + 3s$$

$$x_5 = 55 - 4(2r) - 22(2s) = 55 - 8r - 44s.$$

[Check: if $r=1$, $s=-1$, we get $x_1 = 36$, $x_2 = -6$, $x_5 = 91$ ... confirming (b).]

[12] **2.** The following dictionary arises during the simplex-method solution of a standard problem. All variables are required to be nonnegative.

$$D^0: \quad \begin{array}{rcl} f &=& 12 - 5x_1 + 2x_2 - x_3 + 3x_4 \\ \hline x_5 &=& 20 + 2x_1 - 15x_2 - 7x_3 - 20x_4 \\ x_6 &=& 4 - 4x_1 - 2x_2 + x_3 + 5x_4 \\ x_7 &=& 8 + x_1 + 3x_2 + 11x_3 - 8x_4 \end{array}$$

The questions below all apply to the next step of the method. Answer them *without* completing any simplex pivots or dictionary updates.

(a) Using our notation from class, identify the sets $\mathcal{B}^0$ and $\mathcal{N}^0$ and the BFS $\mathbf{x}^0$ associated with the given dictionary.

(b) List all indices eligible to enter the basis at the next step. ← As in class, use E for Entering, L for Leaving

(c) For each index eligible to enter the basis, list all indices eligible to leave the basis.

(d) For each pair of eligible entering and leaving variables found in (c), determine the updated values of (i) the basis, $\mathcal{B}^+$; (ii) the Basic Feasible Solution (BFS), $\mathbf{x}^+$; and (iii) the objective value, f^+.

(a) $\mathcal{B}^0 = \{5,6,7\}$; $\mathcal{N}^0 = \{1,2,3,4\}$; $\vec{x}^0 = (0,0,0,0,20,4,8)$

(b) $E \in \{2,4\}$ ... indices now in $\mathcal{N}^0$ with ⊕ objective coeffs

(c) If $E=2$, $L=5$ (Limit $x_5 \geq 0$ is stricter than $x_6 \geq 0$ or $x_7 \geq 0$)
If $E=4$, $L \in \{5,7\}$ (Tie between restrictions from $x_5 \geq 0$, $x_7 \geq 0$)

(d) After a pivot, all nonbasics will still be 0. Safely ignore them

• $(E,L)=(2,5)$: $x_2^+ = \frac{20}{15} = \frac{4}{3}$ will push $x_5^+=0$, $x_6^+ = 4-2\left(\frac{4}{3}\right) = \frac{4}{3}$, $x_7^+ = 12$.
$\mathcal{B}^+ = \{2,6,7\}$, $\vec{x}^+ = (0,\frac{4}{3},0,0,0,\frac{4}{3},12)$, $f^+ = 12 + 2\left(\frac{4}{3}\right) = \frac{44}{3}$.

• $(E,L)=(4,5)$: $x_4^+ = \frac{20}{20} = 1$ will push $x_5^+=0$, $x_6^+ = 4+5(1) = 9$, $x_7^+ = 8-8=0$.
$\mathcal{B}^+ = \{4,6,7\}$, $\vec{x}^+ = (0,0,0,1,0,9,0)$, $f^+ = 12+3(1) = 15$.

• $(E,L)=(4,7)$: $x_4^+ = \frac{8}{8} = 1$ will have all same effects just noted.
$\mathcal{B}^+ = \{4,5,6\}$, $\vec{x}^+ = (0,0,0,1,0,9,0)$, $f^+ = 15$.

[In final 2 cases $\mathcal{B}^+$ is degenerate — same $\vec{x}$ for different bases.]

[12] **3.** Use the two-phase method with Anstee's pivoting rules to find all maximizers, or prove that no maximizer exists:

$$\begin{array}{lrl} \text{maximize} & f = x_1 + 3x_2 - 6x_3 & \\ \text{subject to} & x_1 - x_2 - x_3 \leq & 2 \\ & -x_1 \qquad + x_3 \leq & -1 \\ & x_2 - x_3 \leq & 2 \\ & x_1, x_2, x_3 \geq 0 & \end{array}$$

PHASE ONE: $\tilde{f} = -x_0$ replaces f in auxiliary system with dict

$$\begin{array}{l} \tilde{f} = -x_0 \\ \hline w_1 = 2 + x_0 - x_1 + x_2 + x_3 \\ w_2 = -1 + x_0 + x_1 - x_3 \\ w_3 = 2 + x_0 - x_2 + x_3 \end{array}$$

Shrewd initial pivot: x_0 enters, w_2 leaves, via $x_0 = 1 + w_2 - x_1 + x_3$:

$$\tilde{f} = -1 + x_1 - x_3 - w_2$$

$$w_1 = \left.\begin{array}{l} 2 - x_1 + x_2 + x_3 \quad \cancel{+x_0} \\ 1 - x_1 + x_3 + w_2 \end{array}\right\} = 3 - 2x_1 + x_2 + 2x_3 + w_2$$

$$w_3 = \left.\begin{array}{l} 2 - x_2 + x_3 \quad \cancel{+x_0} \\ 1 - x_1 + x_3 + w_2 \end{array}\right\} = 3 - x_1 - x_2 + 2x_3 + w_2$$

New dict:

$$\begin{array}{l} \tilde{f} = -1 + x_1 - x_3 - w_2 \\ w_1 = 3 - 2x_1 + x_2 + 2x_3 + w_2 \\ w_3 = 3 - x_1 - x_2 + 2x_3 + w_2 \\ x_0 = 1 - x_1 + x_3 + w_2 \end{array}$$

Enter x_1, leave x_0: pivot eq^n $x_1 = 1 - x_0 + x_3 + w_2$ gives

$$\tilde{f} = \left.\begin{array}{l} -1 + x_1 - x_3 - w_2 \\ +1 - x_0 + x_3 + w_2 \end{array}\right\} = -x_0$$

$$w_1 = \left.\begin{array}{l} 3 \cancel{-2x_1} + x_2 + 2x_3 + w_2 \\ -2 + 2x_0 - 2x_3 - 2w_2 \end{array}\right\} = 1 + 2x_0 + x_2 - w_2$$

$$w_3 = \left.\begin{array}{l} 3 \cancel{-x_1} - x_2 + 2x_3 + w_2 \\ -1 + x_0 - x_3 - w_2 \end{array}\right\} = 2 + x_0 - x_2 + x_3$$

Continued on page 6

(Blank page for extra calculations.)

Phase One dict:

$$\begin{array}{l} \tilde{f} = 0 - x_0 \\ \hline x_1 = 1 - x_0 + x_3 + w_2 \\ w_1 = 1 + 2x_0 + x_2 - w_2 \\ w_3 = 2 + x_0 - x_2 + x_3 \end{array}$$

Drop x_0 now, b/c $\tilde{f}_{MAX} = 0$, and rebuild dict for original prob.

$$f = x_1 + 3x_2 - 6x_3 = (1 + x_3 + w_2) + 3x_2 - 6x_3.$$

Phase Two Dict:

$$\begin{array}{l} f = 1 + 3x_2 - 5x_3 + w_2 \\ \hline x_1 = 1 + x_3 + w_2 \\ w_1 = 1 + x_2 - w_2 \\ w_3 = 2 - x_2 + x_3 \end{array}$$

Enter x_2, leave w_3, via $x_2 = 2 - w_3 + x_3$:

$$f = 1 \underbrace{+ 3x_2}_{6 - 3w_3 + 3x_3} - 5x_3 + w_2 \Rightarrow$$

$$x_1 = 1 + x_3 + w_2 \Rightarrow$$

$$w_1 = 1 \underbrace{+ x_2}_{2 - w_3 + x_3} - w_2 \Rightarrow$$

$$\begin{array}{l} f = 7 - 3w_3 - 2x_3 + w_2 \\ \hline x_1 = 1 + x_3 + w_2 \\ w_1 = 3 - w_3 + x_3 - w_2 \\ x_2 = 2 - w_3 + x_3 \end{array}$$

Enter w_2, leave w_1, via $w_2 = 3 - w_3 + x_3 - w_1$:

$$f = 7 - 3w_3 - 2x_3 \underbrace{+ w_2}_{3 - w_3 + x_3 - w_1} \Rightarrow$$

$$x_1 = 1 + x_3 \underbrace{+ w_2}_{3 - w_3 + x_3 - w_1} \Rightarrow$$

$$x_2 = 2 - w_3 + x_3$$

$$\begin{array}{l} f = 10 - 4w_3 - x_3 - w_1 \\ \hline x_1 = 4 - w_3 + 2x_3 - w_1 \\ x_2 = 2 - w_3 + x_3 \\ w_2 = 3 - w_3 + x_3 - w_1 \end{array}$$

UNIQUE MAXIMIZER $(x_1^*, x_2^*, x_3^*) = (4, 2, 0)$.

Continued on page 7

[12] **4.** Find two distinct basic feasible solutions that give the maximum value for the following LP:

$$\begin{array}{ll} \text{maximize} & f = -x_1 + 3x_2 + 6x_3 \\ \text{subject to} & 2x_1 - x_2 + x_3 + x_4 \le 60, \\ & 3x_1 + 4x_2 + 2x_3 - 2x_4 \le 150, \\ & x_1, x_2, x_3, x_4 \ge 0. \end{array}$$

Invent slack vars w_1, w_2 to set up a dict ready for Phase Two:

$$\begin{array}{l} f = -x_1 + 3x_2 + 6x_3 \\ \hline w_1 = 60 - 2x_1 + x_2 - x_3 - x_4 \\ w_2 = 150 - 3x_1 - 4x_2 - 2x_3 + 2x_4 \end{array}$$

Enter x_3, leave w_1, via $x_3 = 60 - 2x_1 + x_2 - w_1 - x_4$:

$$f = -x_1 + 3x_2 \cancel{+6x_3} \quad \left.\begin{matrix} \\ 360 - 12x_1 + 6x_2 - 6w_1 - 6x_4 \end{matrix}\right\} \Rightarrow$$

$$w_2 = 150 - 3x_1 - 4x_2 \cancel{-2x_3} + 2x_4 \quad \left.\begin{matrix} \\ -120 + 4x_1 - 2x_2 + 2w_1 + 2x_4 \end{matrix}\right\} \Rightarrow$$

$$\begin{array}{l} f = 360 - 13x_1 + 9x_2 - 6w_1 - 6x_4 \\ \hline x_3 = 60 - 2x_1 + x_2 - w_1 - x_4 \\ w_2 = 30 + x_1 - 6x_2 + 2w_1 + 4x_4 \end{array}$$

Enter x_2, leave w_2, via $x_2 = 5 + \frac{1}{6}x_1 - \frac{1}{6}w_2 + \frac{1}{3}w_1 + \frac{2}{3}x_4$

$$f = 360 - 13x_1 \cancel{+9x_2} - 6w_1 - 6x_4 \quad \left.\begin{matrix} \\ 45 + \frac{3}{2}x_1 - \frac{3}{2}w_2 + 3w_1 + 6x_4 \end{matrix}\right\} \Rightarrow$$

$$x_3 = 60 - 2x_1 \cancel{+x_2} - w_1 - x_4 \quad \left.\begin{matrix} \\ 5 + \frac{1}{6}x_1 - \frac{1}{6}w_2 + \frac{1}{3}w_1 + \frac{2}{3}x_4 \end{matrix}\right\} \Rightarrow$$

$$\begin{array}{l} f = 405 - \frac{23}{2}x_1 - \frac{3}{2}w_2 - 3w_1 \\ \hline x_2 = 5 + \frac{1}{6}x_1 - \frac{1}{6}w_2 + \frac{1}{3}w_1 + \frac{2}{3}x_4 \\ x_3 = 65 - \frac{11}{6}x_1 - \frac{1}{6}w_2 - \frac{2}{3}w_1 - \frac{1}{3}x_4 \end{array}$$

This optimal dict has $f_{MAX} = 405$ at BFS $\vec{x} = (0, 5, 65, 0)$.

But pivoting x_4 into basis and x_3 out also respects optimality. This step would make $x_4^+ = 195$, $x_3^+ = 0$, $x_2^+ = 135$. So a second optimal BFS is $\vec{x} = (0, 135, 0, 195)$.

Continued on page 8

[12] **5.** Consider the following LP, in which a constant parameter k appears:

$$\begin{array}{ll} \text{maximize} & f = -3x_1 + 2x_2 - kx_3 + x_4 \\ \text{subject to} & 2x_1 - 3x_2 - x_3 + x_4 \le 1 \\ & -x_1 + x_2 + 2x_3 - 2x_4 \le 2 \\ & -x_1 + x_2 - 4x_3 + x_4 \le 8 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{array}$$

(a) Show that the problem has a maximum when $k = 10$, and find a point that achieves it.

(b) Show that the problem is unbounded when $k = 0$ and find a feasible $\mathbf{x}$ where $f(\mathbf{x}) > 10^{100}$.

(c) Find the value of k in the interval $(0, 10)$ where the problem's status changes from "has a maximum" to "is unbounded". Describe, as completely as possible, the status of the problem for this k-value. (Does it have a solution? If so, what are the maximizing inputs? Etc.)

(a) Standard Setup for case $k = 10$:

$$\begin{array}{l} f = -3x_1 + 2x_2 - 10x_3 + x_4 \\ \hline w_1 = 1 - 2x_1 + 3x_2 + x_3 - x_4 \\ w_2 = 2 + x_1 - x_2 - 2x_3 + 2x_4 \\ w_3 = 8 + x_1 - x_2 + 4x_3 - x_4 \end{array}$$

Enter x_2, leave w_2, via $x_2 = 2 + x_1 - w_2 - 2x_3 + 2x_4$

Next

$$f = -3x_1 \cancel{+2x_2} - 10x_3 + x_4, \quad 4 + 2x_1 - 2w_2 - 4x_3 + 4x_4 \Rightarrow$$

$$w_1 = 1 - 2x_1 \cancel{+3x_2} + x_3 - x_4, \quad 6 + 3x_1 - 3w_2 - 6x_3 + 6x_4 \Rightarrow$$

$$w_3 = 8 + x_1 \cancel{-x_2} + 4x_3 - x_4, \quad -2 - x_1 + w_2 + 2x_3 - 2x_4 \Rightarrow$$

$$\begin{array}{l} f = 4 - x_1 - 2w_2 - 14x_3 + 5x_4 \\ \hline x_2 = 2 + x_1 - w_2 - 2x_3 + 2x_4 \\ w_1 = 7 + x_1 - 3w_2 - 5x_3 + 5x_4 \\ w_3 = 6 + w_2 + 6x_3 - 3x_4 \end{array}$$

Enter x_4, leave w_3, via $x_4 = 2 + \frac{1}{3}w_2 + 2x_3 - \frac{1}{3}w_3$:

Next

$$f = 4 - x_1 - 2w_2 - 14x_3 \cancel{+5x_4}, \quad 10 + \tfrac{5}{3}w_2 + 10x_3 - \tfrac{5}{3}w_3 \Rightarrow$$

$$x_2 = 2 + x_1 - w_2 - 2x_3 \cancel{+2x_4}, \quad 4 + \tfrac{2}{3}w_2 + 4x_3 - \tfrac{2}{3}w_3 \Rightarrow$$

$$w_1 = 7 + x_1 - 3w_2 - 5x_3 \cancel{+5x_4}, \quad 10 + \tfrac{5}{3}w_2 + 10x_3 - \tfrac{5}{3}w_3 \Rightarrow$$

$$\begin{array}{l} f = 14 - x_1 - \frac{1}{3}w_2 - 4x_3 - \frac{5}{3}w_3 \\ \hline x_2 = 6 + x_1 - \frac{1}{3}w_2 + 2x_3 - \frac{2}{3}w_3 \\ x_4 = 2 + \frac{1}{3}w_2 + 2x_3 - \frac{1}{3}w_3 \\ w_1 = 17 + x_1 - \frac{4}{3}w_2 + 5x_3 - \frac{5}{3}w_3 \end{array}$$

UNIQUE MAXIMIZER AT $\vec{x} = (0, 6, 0, 2)$.

(Blank page for extra calculations.)

(b) If $k=0$ instead, then objective row above would be

$$f = -3x_1 \boxed{+2x_2} + x_4$$

$$12 + 2x_1 - \tfrac{2}{3}w_2 + 4x_3 - \tfrac{4}{3}w_3$$

$$2 \qquad + \tfrac{1}{3}w_2 + 2x_3 - \tfrac{1}{3}w_3$$

$$= 14 - x_1 - \tfrac{1}{3}w_2 + 6x_3 - \tfrac{5}{3}w_3$$

Thus x_3 is eligible to enter, but there is no upper limit on the possible value for x_3. For every $t \geq 0$, the point $\vec{x}(t) = (0, 6+2t, t, 2+2t)$ is feasible, with $f(\vec{x}(t)) = 14+6t$. Choose any $t > \frac{1}{6}(10^{100} - 14)$ here to get $f(\vec{x}(t)) > 10^{100}$.

(c) For general k, add "$-kx_3$" to the result from $k=0$ in (b):

$$f = 14 - x_1 - \tfrac{1}{3}w_2 + (6-k)x_3 - \tfrac{5}{3}w_3.$$

- If $k>6$, all coeffs here are negative, so we keep the unique maximizer at $(0,6,0,2)$ with $f_{MAX} = 14$.
- If $k<6$, coeff of x_3 is positive and problem is unbounded, much as in (b) above.
- If $k=6$, the transitional value, there is a massive tie for the value $f_{MAX}=14$, with the pts $\vec{x}(t) = (0, 6+2t, t, 2+2t)$, $t \geq 0$, found in (b) for related reasons.

The End